1. Express $\left(\frac{-\sqrt{3}}{2}+\frac{i}{2}\right)^{603}$ in the form $a+ib$. Simplify your answer as much as possible. (It may be convenient to express $\frac{-\sqrt{3}}{2}+\frac{i}{2}$ in polar form.)

We first write $\frac{-\sqrt{3}}{2}+\frac{i}{2}$ in polar form. The modulus of $\frac{-\sqrt{3}}{2}+\frac{i}{2}$ is

$$\left|\frac{-\sqrt{3}}{2}+\frac{i}{2}\right| = \sqrt{\left(\frac{-\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1.$$

Therefore, the argument of $\frac{-\sqrt{3}}{2}+\frac{i}{2}$ will be given by a θ satisfying

$$e^{i\theta} = \cos\theta + i\sin\theta = \frac{-\sqrt{3}}{2}+\frac{i}{2}$$

which means $\theta = 5\pi/6$ (or $\theta = 5\pi/6 + 2k\pi$ for $k \in \mathbb{Z}$). We then have

$$\begin{aligned}\left(\frac{-\sqrt{3}}{2}+\frac{i}{2}\right)^{603} &= \left(e^{i5\pi/6}\right)^{603} \\ &= e^{i5\pi 603/6} \\ &= e^{i5\pi(100+1/2)} \\ &= e^{i500\pi}e^{i5\pi/2} \\ &= 1e^{i2\pi+i\pi/2} \\ &= e^{i\pi/2} = i.\end{aligned}$$

Therefore

$$\left(\frac{-\sqrt{3}}{2}+\frac{i}{2}\right)^{603} = i.$$

2. (a) State all solutions to the equation $z^7 = 1$. (You may state your answer in any convenient form.)

Writing $z = re^{i\theta}$ the equation becomes

$$\begin{aligned}(re^{i\theta})^7 &= 1 = e^{i2k\pi}\\ r^7 e^{i7\theta} &= e^{i2k\pi}\end{aligned}$$

so $r^7 = 1$ which implies $r = 1$, and $7\theta = 2k\pi$ which implies $\theta = 2k\pi/7$. Thus the solutions are

$$z = e^{i2k\pi/7} = \cos 2k\pi/7 + i \sin 2k\pi/7$$

where $k \in \{0, 1, 2, 3, 4, 5, 6\}$ (or any other 7 consecutive integers).[1]

(b) Find all complex numbers z satisfying the equation

$$(z-1)^7 = (z+2)^7$$

(you may state your answer in terms of the solutions to part (a) if you wish).

We first observe that $z = -2$ is not a solution since substituting in $z = -2$ in both sides leads to $(-3)^7 = 0$ which is false. We can thus divide both sides of the equation by $(z+2)^7$ to find that

$$\left(\frac{z-1}{z+2}\right)^7 = 1.$$

This means that $w = \frac{z-1}{z+2}$ is a solution to $w^7 = 1$ so, using part (a), we must therefore have

$$\frac{z-1}{z+2} = e^{i2\pi k/7} \tag{1}$$

with $k \in \{0, 1, 2, 3, 4, 5, 6\}$. Solving for z leads to

$$z - 1 = e^{i2\pi k/7}(z+2) \quad \Longleftrightarrow \quad (1 - e^{i2\pi k/7})z = 1 + 2e^{i2\pi k/7}$$

which leads us to conclude that $z = \frac{1+2e^{i2\pi k/7}}{1-e^{i2\pi k/7}}$ where $k \in \{1, 2, 3, 4, 5, 6\}$. (Notice that we don't get a solution for $k = 0$ because then we would be dividing by 0. In that case, equation (1) above becomes $\frac{z-1}{z+2} = 1$ which has no solutions.)[2]

[1] Note that since I said state the solutions you would get full credit for stating them correctly without any sort of derivation.

[2] Can you come up with a good explanation for why this equation only has 6 solutions even though it appears to be a 7-th order equation?

3. For each of the following, identify the largest open disk on which the series converges. Justify your answer.

   (a) $\sum_{n=0}^{\infty}[1+(-1)^n]^n z^n$

   According to the root test, the radius of convergence will be given by $1/\limsup|a_n|^{1/n}$ with $a_n = [1+(-1)^n]^n$. We have that

$$|a_n|^{1/n} = |[1+(-1)^n]^n|^{1/n} = 1+(-1)^n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

   We have therefore have for any $n \in \mathbb{N}$ that

$$\sup_{k\geq n}|a_k|^{1/k} = \sup\{0,2\} = 2$$

   so

$$\limsup|a_n|^{1/n} := \lim_{n\to\infty}\sup_{k\geq n}|a_k|^{1/k} = \lim_{n\to\infty} 2 = 2.$$

   Therefore the radius of convergence of $\sum_{n=0}^{\infty}[1+(-1)^n]^n z^n$ is $1/2$, so the (open) disk of convergence is the set of $z \in \mathbb{C}$ with $|z| < 1/2$.

   (b) $\sum_{n=0}^{\infty}(3z+6)^n$

   Solution 1: We make the substitution $w = (3z+6)$ so the series becomes $\sum_{n=0}^{\infty} w^n$. This series has radius of convergence 1 (using either that we proved in class that $\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}$ for $|w| < 1$ or using the root test: $\lim_{n\to\infty} 1^{1/n} = 1$). Therefore the largest open disk on which the series converges is $|w| < 1$ which is equivalent to $|3z+6| < 1$ and hence $|z+2| < 1/3$. We conclude that largest open disk on which $\sum_{n=0}^{\infty}(3z+6)^n$ converges has radius $1/3$ and center $z = -2$.

   Solution 2: We rewrite the series

$$\sum_{n=0}^{\infty}(3z+6)^n = \sum_{n=0}^{\infty} 3^n(z+2)^n.$$

   We compute

$$\lim_{n\to\infty}|3^n|^{1/n} = \lim_{n\to\infty} 3 = 3$$

   so according to the root test, the radius of convergence is $1/3$. Since this is a power series in $z+2 = z-(-2)$ the center of the disk of convergence is $z = -2$. Therefore, the largest open disk on which the series converges is the set of $z \in \mathbb{C}$ with $|z+2| < 1/3$, i.e. the disk of radius $1/3$ centered at $z = -2$.

4. (a) Consider functions u, $v : U \subset \mathbb{C} \to \mathbb{R}$, and assume that the partial derivatives of u and v exist on U. If $f(x+iy) = u(x,y) + iv(x,y)$, state what it means for f to satisfy the Cauchy-Riemann equations on U. (You may write the equations in terms of f or in terms of u and v)

To say that f satisfies the Cauchy-Riemann equations on U means that

$$f_y(z) = i f_x(z) \text{ for all } z = x + iy \in U$$

or equivalently that

$$u_x(z) = v_y(z) \text{ and } u_y(z) = -v_x(z) \text{ for all } z = x + iy \in U.$$

(b) Find all possible functions $v : \mathbb{C} \to \mathbb{R}$ for which

$$f(x,y) = x^4 - 6x^2y^2 + y^4 + y + iv(x,y)$$

is an analytic function, or prove that no such v exists.

We need to find $v(x,y)$ so that f satisfies the Cauch-Riemann equations, or prove that finding such a v is not possible. With $u(x,y) = x^4 - 6x^2y^2 + y^4 + y$, the equation $u_x = v_y$ leads us to

$$v_y(x,y) = 4x^3 - 12xy^2$$

from which we can conclude that

$$v(x,y) = y\text{-antiderivative of } (4x^3 - 12xy^2) = 4x^3y - 4xy^3 + h(x)$$

where $h(x)$ is a differentiable function depending only on x. From this we can conclude that

$$v_x = 12x^2y - 4y^3 + h'(x)$$

which when used with the equation $u_y = -v_x$ leads us to conclude that

$$-12x^2y + 4y^3 + 1 = -12x^2y + 4y^3 - h'(x)$$

so $h'(x) = -1$ and hence $h'(x) = -x + c$ for some constant c. We've thus see that f satisfies the Cauchy-Riemann equations if and only if v is given by

$$v(x,y) = 4x^3y - 4xy^3 - x + c$$

for some constant c.

5. Recall that $\cos z$ for $z \in \mathbb{C}$ is defined by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Find all complex numbers z satisfying the equation $\cos z = 3$. (Hint: First solve for e^{iz}.)

Using the definition of $\cos z$ we have that

$$\frac{e^{iz} + e^{-iz}}{2} = 3$$

and multiplying both sides by $2e^{iz}$ (which doesn't change the solution set because e^{iz} is never 0) we find that

$$(e^{iz})^2 + 1 = 6e^{iz}$$

or equivalently

$$(e^{iz})^2 - 6e^{iz} + 1 = 0.$$

This is a quadratic equation in e^{iz}, so using the quadratic formula, we find that

$$e^{iz} = \frac{6 \pm \sqrt{(6)^2 - 4}}{2} = 3 \pm \tfrac{1}{2}\sqrt{32} = 3 \pm 2\sqrt{2}.$$

Notice that $3 - 2\sqrt{2}$ is positive since $3^2 = 9$ while $(2\sqrt{2})^2 = 8$. To solve $e^{iz} = 3 \pm 2\sqrt{2}$ we write

$$e^{iz} = e^{i(x+iy)} = e^{-y+ix} = e^{-y}e^{ix} \tag{2}$$

while writing $3 \pm 2\sqrt{2} > 0$ in polar form gives

$$3 \pm 2\sqrt{2} = e^{\log(3\pm 2\sqrt{2})}e^{i0} = e^{\log(3\pm 2\sqrt{2})}e^{i2\pi k} \tag{3}$$

for $k \in \mathbb{Z}$. Comparing (2) and (3) leads us to conclude that

$$y = -\log(3 \pm 2\sqrt{2}) \text{ and } x = 2\pi k$$

so the solutions to $\cos z = 3$ are

$$z = 2\pi k - i\log(3 \pm 2\sqrt{2})$$

for any $k \in \mathbb{Z}$.