Suprema/Infima Review Sheet

This is a review of some basic facts about suprema and infima. You may use these facts in your homework assignments without proving them.

Definition 1. Consider a set $A \subset \mathbb{R}$.

(1) A number u is said to be an *upper bound* for A if $u \geq a$ for all $a \in A$.
(2) A number l is said to be an *lower bound* for A if $l \leq a$ for all $a \in A$.
(3) The set A is said to be *bounded above* if there exists an upper bound for A.
(4) The set A is said to be *bounded below* if there exist a lower bound for A.
(5) The set A is said to be *bounded* if A is bounded above and bounded below.
(6) A number s is said to be a *supremum* or *least upper bound* for A if:
    (a) s is an upper bound for A, and
    (b) if u is an upper bound for A, then $u \geq s$ (i.e. s is less than or equal to every other upper bound for A).
(7) A number i is said to be an *infimum* or *greatest lower bound* for A if:
    (a) i is a lower bound for A, and
    (b) if l is an upper bound for A, then $l \leq i$ (i.e. i is greater than or equal to every other lower bound for A).

Remark 2. It is a straightforward consequence of the definition of supremum that a set can have at most one supremum. If a set A has a supremum, we denote it by $\sup A$. Similarly, a set can have at most one infimum. If a set A has an infimum, we denote it by $\inf A$.

The Completeness Axiom. Every nonempty set of real numbers which is bounded above has a supremum

The following theorem is logically equivalent to the Completeness Axiom.

Theorem 3. Every nonempty set of real numbers which is bounded below has an infimum.

Theorem 4 (Approximation property of sup/inf)**.** Consider a set $A \subset \mathbb{R}$, and assume that A is not empty.

(1) Assume that A is bounded above, so that $\sup A$ exists. For any $\varepsilon > 0$, there is an element $a \in A$ (which in general depends on ε) satisfying
$$a > \sup A - \varepsilon.$$
(2) Assume that A is bounded below, so that $\inf A$ exists. For any $\varepsilon > 0$, there is an element $a \in A$ (which in general depends on ε) satisfying
$$a < \inf A + \varepsilon.$$

Remark 5. According to the Competeness Axiom and Theorem 3, a set may fail to have a supremum or infimum in the event that it is not bounded, or is empty. We will extend the definitions of sup and inf to these cases by using the following conventions:

(1) If A is not bounded above, we define $\sup A = +\infty$.
(2) If A is not bounded below, we define $\inf A = -\infty$.
(3) For the empty set we define $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

With these conventions, every set of real numbers has a supremum and an infimum, but one or both may be infinite. Note that Theorem 4 only applies when the sup or inf is finite.

Definition 6. Let A and B be nonempty sets of real numbers, and let $c \in \mathbb{R}$ be a constant.

(1) The set $A + B$ is defined by
$$A + B := \{a + b \,|\, a \in A \text{ and } b \in B\}.$$

(2) The set cA is defined by
$$cA := \{ca \,|\, a \in A\}.$$

Theorem 7 (Some properties of sup's and inf's)**.** Let A and B be sets of real numbers. Then:

(1) If $A \subseteq B$ then
$$\sup A \leq \sup B$$
and
$$\inf A \geq \inf B.$$

(2) If A is nonempty, then
$$\inf A \leq \sup A.$$

(3) If A and B are nonempty then[1]
$$\inf(A + B) = \inf A + \inf B$$
and
$$\sup(A + B) = \sup A + \sup B.$$

(4) If A is nonempty and $c \in \mathbb{R}$ is a constant, then[2]
$$\inf(cA) = \begin{cases} c \inf A & \text{if } c \geq 0 \\ c \sup A & \text{if } c < 0 \end{cases}$$
and
$$\sup(cA) = \begin{cases} c \sup A & \text{if } c \geq 0 \\ c \inf A & \text{if } c < 0. \end{cases}$$

[1] If either of the sets A or B is not bounded, these identities remain true as long as we use the conventions $+\infty + (\text{constant}) = +\infty$, $-\infty + (\text{constant}) = -\infty$, $(+\infty) + (+\infty) = +\infty$, and $(-\infty) + (-\infty) = -\infty$. Note that as long as both sets are nonempty, we would never have occasion to add $+\infty$ and $-\infty$.

[2] If A is not bounded, these identities remain true as long as we use the conventions
$$c \cdot (+\infty) = \begin{cases} +\infty & \text{if } c > 0 \\ -\infty & \text{if } c < 0 \\ 0 & \text{if } c = 0 \end{cases}$$
and
$$c \cdot (-\infty) = \begin{cases} -\infty & \text{if } c > 0 \\ +\infty & \text{if } c < 0 \\ 0 & \text{if } c = 0. \end{cases}$$