Review Sheet: Sequences in $\mathbb{R}$

Definition 1. Let S be any set. A *sequence in* S is a function $f : \mathbb{N} \to S$.

Usually we will denote $x_k = f(k)$. Common notations for sequences include: $\{x_k\}$, (x_k),$\{x_k\}_{k=1}^{\infty}$, $(x_k)_{k=1}^{\infty}$, $\{x_k\}_{k\in\mathbb{N}}$, $(x_k)_{k\in\mathbb{N}}$, $\{x_1, x_2, \dots\}$, $(x_1, x_2, \dots)$.

Definition 2. Let $\{x_k\}$ be a sequence in $\mathbb{R}$.

(1) $\{x_k\}$ is said to *converge to* $L \in \mathbb{R}$ if for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ so that for all $k \geq N$, $|x_k - L| < \varepsilon$. If $\{x_k\}$ converges to L, we will write $\lim_{k\to\infty} x_k = L$, and L is said to be the *limit* of $\{x_k\}$.
(2) $\{x_k\}$ is said to be *convergent* if x_k converges to some $L \in \mathbb{R}$.
(3) $\{x_k\}$ is said to be *bounded* if there is an $M > 0$ so that $|x_k| \leq M$ for all $k \in \mathbb{N}$.
(4) $\{x_k\}$ is said to be a *Cauchy sequence* if for any $\varepsilon > 0$ there an is an $N \in \mathbb{N}$ so that for any $n \geq N$ and $m \geq N$, $|x_n - x_m| < \varepsilon$.
(5) $\{x_k\}$ is said to be *increasing* if $x_{k+1} \geq x_k$ for all $k \in \mathbb{N}$.
(6) $\{x_k\}$ is said to be *strictly increasing* if $x_{k+1} > x_k$ for all $k \in \mathbb{N}$.
(7) $\{x_k\}$ is said to be *decreasing* if $x_{k+1} \leq x_k$ for all $k \in \mathbb{N}$.
(8) $\{x_k\}$ is said to be *strictly decreasing* if $x_{k+1} < x_k$ for all $k \in \mathbb{N}$.
(9) $\{x_k\}$ is said to be *monotonic* if it is either decreasing or increasing.
(10) A *subsequence of* $\{x_k\}$ is a sequence of the form $\{x_{j_k}\}_{k\in\mathbb{N}}$, where $\{j_k\}_{k\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$.

It is a straightforward consequence of the definition of convergence that a sequence in $\mathbb{R}$ can have at most one limit (which is why we say "*the* limit of $\{x_k\}$" above, rather than "*a* limit of $\{x_k\}$"). Some other basic consequences of the definitions above are given in Theorems 3, 4 and 5.

Theorem 3. Let $\{x_k\}$ be a convergent sequence in $\mathbb{R}$. Then:

(1) $\{x_k\}$ is bounded.
(2) $\{x_k\}$ is a Cauchy sequence.
(3) Every subsequence $\{x_{j_k}\}$ of $\{x_k\}$ is convergent and has the same limit as $\{x_k\}$.

Theorem 4. Let $\{x_k\}$ and $\{y_k\}$ be sequences in $\mathbb{R}$, and assume that $\{x_k\}$ converges to $L \in \mathbb{R}$ and that $\{y_k\}$ converges to $M \in \mathbb{R}$. Then:

(1) $\{x_k + y_k\}$ converges to $L + M$.
(2) $\{x_k y_k\}$ converges to LM.
(3) If $M \neq 0$ then $\left\{\frac{x_k}{y_k}\right\}$ converges to $\frac{L}{M}$.
(4) If $x_k \geq y_k$ for all $k \in \mathbb{N}$, then $L \geq M$.

Theorem 5 (The Squeeze Theorem)**.** Consider sequences $\{x_k\}$, $\{y_k\}$, and $\{z_k\}$ satisfying

$$x_k \leq y_k \leq z_k \qquad \text{for all } k \in \mathbb{N}.$$

If $\{x_k\}$ and $\{z_k\}$ are convergent and satisfy $\lim_{k\to\infty} x_k = \lim_{k\to\infty} z_k = L$ then $\{y_k\}$ is convergent and $\lim_{k\to\infty} y_k = L$.

Theorems 6, 7, and 8 are each logically equivalent to the completeness axiom for real numbers.

Theorem 6 (Monotone Convergence Theorem for Sequences)**.** Bounded, monotonic sequences in $\mathbb{R}$ are convergent.

Theorem 7 (Bolzano-Weierstrass Theorem)**.** A bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Theorem 8 (Cauchy Criterion for Covergence)**.** A Cauchy sequence in $\mathbb{R}$ is convergent.