Worksheet on Images/Preimages

Definition 1. Let A and B be any sets, and let $f : A \to B$ be a function.

(1) Let C be a subset of the domain A of f. The *image* of C under f is the set $f(C)$ defined by

$$f(C) := \{y \in B \,|\, \exists x \in C \text{ with } f(x) = y\} = \bigcup_{x \in C} \{f(x)\}$$

(2) Let D be a subset of the codomain B of f. The *preimage* of D under f is the set $f^{-1}(D)$ defined by[1]

$$f^{-1}(D) := \{x \in A \,|\, f(x) \in D\},$$

that is

$$x \in f^{-1}(D) \iff f(x) \in D.$$

More informally, the image of C under f is the set of values that f takes on C, while the preimage of D under f is the set of points in the domain mapped to D by the function.

Here are some exercises to help understand these concepts. See section 1.5 in the textbook for help.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ find each of the following images/preimages:
   (a) $f([-1, 2)) = ?$
   (b) $f([-2, 1) \cup (3, 5)) = ?$
   (c) $f^{-1}([-1, 1] \cup \{10\}) = ?$
   (d) $f^{-1}(\{1\} \cup [9, 16)) = ?$
   (e) $f^{-1}([-2, -1]) = ?$

(2) Consider a function $f : A \to B$.
   (a) If $\{C_\alpha\}_{\alpha \in I}$ is a collection of subsets of A, prove that
   $$f\left(\bigcup_{\alpha \in I} C_\alpha\right) = \bigcup_{\alpha \in I} f(C_\alpha).$$
   (b) If $\{D_\alpha\}_{\alpha \in I}$ is a collection of subsets of B, prove that
   $$f^{-1}\left(\bigcup_{\alpha \in I} D_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(D_\alpha).$$
   (c) If $\{C_\alpha\}_{\alpha \in I}$ is a collection of subsets of A, prove that
   $$f\left(\bigcap_{\alpha \in I} C_\alpha\right) \subset \bigcap_{\alpha \in I} f(C_\alpha),$$
   and find an example that shows we can't replace the "$\subset$" with an "$=$".
   (d) If $\{D_\alpha\}_{\alpha \in I}$ is a collection of subsets of B, prove that
   $$f^{-1}\left(\bigcap_{\alpha \in I} D_\alpha\right) = \bigcap_{\alpha \in I} f^{-1}(D_\alpha).$$
   (e) If C_1 and C_2 are subsets of A, show that
   $$f(C_1 \setminus C_2) \supset f(C_1) \setminus f(C_2)$$
   and find an example that shows we can't replace the "$\supset$" with an "$=$".
   (f) If D_1 and D_2 are subsets of B, show that
   $$f^{-1}(D_1 \setminus D_2) = f^{-1}(D_1) \setminus f^{-1}(D_2).$$

[1] The term "*inverse image*" is sometimes used to mean the same thing as *preimage*. Do not let the word "inverse" or the notation $f^{-1}(D)$ confuse you into thinking the function f in question is invertible. The preimage $f^{-1}(D)$ makes sense for any function $f : A \to B$ whether there exists an inverse function $f^{-1} : B \to A$ or not.