Math 421, Homework #9 Solutions

(1) (a) A set $E \subset \mathbb{R}^n$ is said to be *path connected* if for any pair of points $\mathbf{x} \in E$ and $\mathbf{y} \in E$ there exists a continuous function $\boldsymbol{\gamma} : [0,1] \to \mathbb{R}^n$ satisfying $\boldsymbol{\gamma}(0) = \mathbf{x}$, $\boldsymbol{\gamma}(1) = \mathbf{y}$, and $\boldsymbol{\gamma}(t) \in E$ for all $t \in [0,1]$. Let $E \subset \mathbb{R}^n$ and assume that E is path connected. Prove that E is connected.

Proof. We will argue by contradiction. Assume that E is not connected. Then there exists sets $U \subset E$ and $V \subset E$ which are nonempty, disjoint ($U \cap V = \emptyset$), relatively open in E, and $U \cup V = E$.

Consider points $\mathbf{x} \in U$ and $\mathbf{y} \in V$ (which we can do because we assume that U and V are both nonempty). Since E is path connected, we can find a continuous map $\boldsymbol{\gamma} : [0,1] \to \mathbb{R}^n$ satisfying $\boldsymbol{\gamma}(0) = \mathbf{x}$, $\boldsymbol{\gamma}(1) = \mathbf{y}$, and $\boldsymbol{\gamma}([0,1]) \subset E$. Define sets $U' = U \cap \boldsymbol{\gamma}([0,1])$ and $V' = V \cap \boldsymbol{\gamma}([0,1])$. We claim that U' and V' separate $\boldsymbol{\gamma}([0,1])$. Indeed, since $\mathbf{x} \in U$ and $\mathbf{x} = \boldsymbol{\gamma}(0) \in \boldsymbol{\gamma}([0,1])$ we have that $\mathbf{x} \in U' = U \cap \boldsymbol{\gamma}([0,1])$ so U' is nonempty. Similarly $\mathbf{y} \in V' = V \cap \boldsymbol{\gamma}([0,1])$ so V' is nonempty. Since $U \cap V = \emptyset$ we have that

$$U' \cap V' = (U \cap \boldsymbol{\gamma}([0,1])) \cap (V \cap \boldsymbol{\gamma}([0,1])) = U \cap V \cap \boldsymbol{\gamma}([0,1]) = \emptyset,$$

and similarly, since $\boldsymbol{\gamma}([0,1]) \subset E$ by assumption we have that

$$\begin{aligned} U' \cup V' &= (U \cap \boldsymbol{\gamma}([0,1])) \cup (V \cap \boldsymbol{\gamma}([0,1])) \\ &= (U \cup V) \cap \boldsymbol{\gamma}([0,1]) \\ &= E \cap \boldsymbol{\gamma}([0,1]) = \boldsymbol{\gamma}([0,1]). \end{aligned}$$

Finally, we observe that the Lemma stated in the solutions to Homework #5 implies that U' and V' are relatively open in $\boldsymbol{\gamma}([0,1])$. We conclude that $\boldsymbol{\gamma}([0,1])$ is not connected.

However, since $[0,1]$ is connected, and $\boldsymbol{\gamma}$ is continuous, it follows from Theorem 9.30 that $\boldsymbol{\gamma}([0,1])$ is connected. This contradiction shows that E must be connected as well. □

(b) Prove that open balls in $\mathbb{R}^n$ are connected, i.e. given $\mathbf{a} \in \mathbb{R}^n$ and $r > 0$ prove that $B_r(\mathbf{a})$ is connected.

Proof. Applying part (a) it suffices to show that an open ball is always path connected. Let $\mathbf{a} \in \mathbb{R}^n$, let $r > 0$, and let $\mathbf{x}$, $\mathbf{y} \in B_r(\mathbf{a})$.

Define $\boldsymbol{\gamma} : [0,1] \to \mathbb{R}^n$ by

$$\boldsymbol{\gamma}(t) = (1-t)\mathbf{x} + t\mathbf{y}.$$

Then $\boldsymbol{\gamma}$ is a continuous function since each component function is a first-order polynomial in t. Moreover $\boldsymbol{\gamma}(0) = \mathbf{x}$ and $\boldsymbol{\gamma}(1) = \mathbf{y}$. We claim that $\boldsymbol{\gamma}(t) \in B_r(\mathbf{a})$ for all $t \in [0,1]$. Indeed for $t = 0$ or $t = 1$ we have that $\boldsymbol{\gamma}(t) \in B_r(\mathbf{a})$ by assumption. Moreover, for $t \in (0,1)$ we have that

$$\begin{aligned} \|\boldsymbol{\gamma}(t) - \mathbf{a}\| &= \|(1-t)\mathbf{x} + t\mathbf{y} - \mathbf{a}\| \\ &= \|(1-t)(\mathbf{x} - \mathbf{a}) + t(\mathbf{y} - \mathbf{a})\| && \mathbf{a} = (1-t)\mathbf{a} + t\mathbf{a} \\ &\le \|(1-t)(\mathbf{x} - \mathbf{a})\| + \|t(\mathbf{y} - \mathbf{a})\| && \text{triangle inequality} \\ &= |1-t|\,\|\mathbf{x} - \mathbf{a}\| + |t|\,\|\mathbf{y} - \mathbf{a}\| && \text{properties of norms} \\ &< |1-t|\,r + |t|\,r && \mathbf{x},\ \mathbf{y} \in B_r(\mathbf{a}),\ t \neq 0 \text{ and } t \neq 1 \\ &= (1-t)r + tr = r && \text{since } t \in (0,1). \end{aligned}$$

We conclude that $\boldsymbol{\gamma}(t) \in B_r(\mathbf{a})$ for all $t \in [0,1]$. Therefore $B_r(\mathbf{a})$ is path-connected, and by part (a), $B_r(\mathbf{a})$ is connected. □

(c) Prove that $\mathbb{R}^n$ is connected.

Proof. Arguing as in part (b), given $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$, the function $\boldsymbol{\gamma} : [0,1] \to \mathbb{R}^n$ defined by

$$\boldsymbol{\gamma}(t) = (1-t)\mathbf{x} + t\mathbf{y}$$

is a continuous function with $\boldsymbol{\gamma}(0) = \mathbf{x}$, $\boldsymbol{\gamma}(1) = \mathbf{y}$ and $\boldsymbol{\gamma}(t) \in \mathbb{R}^n$ for all $t \in [0,1]$. Therefore $\mathbb{R}^n$ is path connected, and hence connected by part (a). $\square$

Alternate proof. We have that

$$\mathbb{R}^n = \bigcup_{k\in\mathbb{N}} B_k(\mathbf{0})$$

and that

$$\bigcap_{k\in\mathbb{N}} B_k(\mathbf{0}) = B_1(\mathbf{0}) \neq \emptyset.$$

Therefore problem 2(b) from Homework #5 tells us that $\mathbb{R}^n$ is connected since each of the sets $B_k(\mathbf{0})$ is connected. $\square$

(d) Prove that only subsets of $\mathbb{R}^n$ which are both open and closed are $\mathbb{R}^n$ and $\emptyset$.

Proof. Assume that $U \subset \mathbb{R}^n$ is open and closed, and that $U \neq \mathbb{R}^n$ and $U \neq \emptyset$. We claim that U and $V := U^c$ separate $\mathbb{R}^n$. Indeed U is open (and hence relatively open in $\mathbb{R}^n$) and nonempty by assumption. Since U is also assumed to be closed, $V = U^c$ is also open. Moreover, since we assume that $U \neq \mathbb{R}^n$ it must by that $V = \mathbb{R}^n \setminus U$ is nonempty. Finally we have that

$$U \cap V = U \cap U^c = \emptyset$$

and that

$$U \cup V = U \cup U^c = \mathbb{R}^n.$$

Therefore U and V separate $\mathbb{R}^n$ and therefore $\mathbb{R}^n$ is not connected in contradiction part to (c). This contradiction lets us conclude that any subset U of $\mathbb{R}^n$ that is both open and closed must satisfy either $U = \mathbb{R}^n$ or $U = \emptyset$. $\square$

(2) Consider a function $\mathbf{f} : E \subset \mathbb{R}^n \to \mathbb{R}^m$ and assume that $\mathbf{f}$ is continuous at some point $\mathbf{a} \in E$ and that $\mathbf{f}(\mathbf{a}) \neq \mathbf{0}$. Prove that there is an $r > 0$ so that for all $\mathbf{x} \in E$ with $\|\mathbf{x} - \mathbf{a}\| < r$, $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$.

Proof. Since $\mathbf{f}(\mathbf{a}) \neq \mathbf{0}$ and $\mathbf{f}$ is assumed to be continuous at $\mathbf{a}$ we can find an $r > 0$ so that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\| < \frac{1}{2} \|\mathbf{f}(\mathbf{a})\| \, (\neq 0) \qquad \text{for } \mathbf{x} \in E \text{ with } \|\mathbf{x} - \mathbf{a}\| < r.$$

Using the triangle inequality, it follows that for $\mathbf{x} \in E$ with $\|\mathbf{x} - \mathbf{a}\| < r$

$$|\|\mathbf{f}(\mathbf{x})\| - \|\mathbf{f}(\mathbf{a})\|| \leq \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\| < \frac{1}{2} \|\mathbf{f}(\mathbf{a})\|$$

or equivalently that

$$-\frac{1}{2} \|\mathbf{f}(\mathbf{a})\| < \|\mathbf{f}(\mathbf{x})\| - \|\mathbf{f}(\mathbf{a})\| < \frac{1}{2} \|\mathbf{f}(\mathbf{a})\|$$

and hence

$$\|\mathbf{f}(\mathbf{x})\| > \frac{1}{2} \|\mathbf{f}(\mathbf{a})\| \neq 0 \qquad \text{for } \mathbf{x} \in E \text{ with } \|\mathbf{x} - \mathbf{a}\| < r.$$

Therefore $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$ for $\mathbf{x} \in E$ with $\|\mathbf{x} - \mathbf{a}\| < r$. $\square$

Alternate proof. Assume not. Then for every $r > 0$ there is an $\mathbf{x}_r \in E$ satisfying $\|\mathbf{x}_r - \mathbf{a}\| < r$ and $\mathbf{f}(\mathbf{x}_r) = \mathbf{0}$. In particular, for each $k \in \mathbb{N}$ there is an $\mathbf{x}_k \in E$ satisfying $\|\mathbf{x}_k - \mathbf{a}\| < \frac{1}{k}$ and $\mathbf{f}(\mathbf{x}_k) = \mathbf{0}$. But then we can use the squeeze theorem with $\|\mathbf{x}_k - \mathbf{a}\| < \frac{1}{k}$ to conclude that $\mathbf{x}_k \to \mathbf{a}$ so $\mathbf{x}_k$ is a sequence in E converging to $\mathbf{a}$. Consequently, the sequential characterization of continuity and the assumption that $\mathbf{f}$ is continuous at $\mathbf{a}$ let's us conclude that

$$\lim_{k \to \infty} \mathbf{f}(\mathbf{x}_k) = \mathbf{f}(\mathbf{a}).$$

However, since $\mathbf{f}(\mathbf{x}_k) = \mathbf{0}$ for all $k \in \mathbb{N}$, this leads to the contradiction

$$\mathbf{0} \neq \mathbf{f}(\mathbf{a}) = \lim_{k \to \infty} \mathbf{f}(\mathbf{x}_k) = \lim_{k \to \infty} \mathbf{0} = \mathbf{0}.$$

This contradiction allows us to conclude the original statement is true. $\square$

(3) Let $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$ be open intervals and let $(a,b) \in I \times J$. Given a function $f : I \times J \to \mathbb{R}$ define functions $g : I \to \mathbb{R}$ and $h : J \to \mathbb{R}$ by

$$g(x) = f(x,b) \text{ and } h(x) = f(a,x).$$

(a) Assume that f is continuous at (a,b). Prove that g is continuous at a and that h is continuous at b.

Proof. Let $\varepsilon > 0$ Since f is assumed to be continuous at (a,b), there exists a $\delta > 0$ so that

$$|f(x,y) - f(a,b)| < \varepsilon \quad \text{ for } (x,y) \in I \times J \text{ with } \|(x,y) - (a,b)\| < \delta.$$

Then if $x \in I$ and $|x - a| < \delta$, we have $\|(x,b) - (a,b)\| = \sqrt{(x-a)^2} = |x-a| < \delta$ so we can use the definition of g to conclude that

$$|g(x) - g(a)| = |f(x,b) - f(a,b)| < \varepsilon.$$

Therefore g is continuous at a. Similarly, we have that if $y \in J$ and $|y - b| < \delta$, then $\|(a,y) - (a,b)\| = |y - b| < \delta$ so

$$|h(y) - h(b)| = |f(a,y) - f(a,b)| < \varepsilon,$$

and hence h is continuous at b. □

(b) Show that the converse of part (a) is not true, i.e. find an example of a function $f : I \times J \to \mathbb{R}$ which is not continuous at (a,b) but where g is continuous at a and h is continuous at b (with g and h defined as above).

Example. Let $f : (-1,1) \times (-1,1) \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & xy = 0 \\ 1 & xy \neq 0 \end{cases}$$

and let $(a,b) = (0,0)$. Then it's easy to show that f is not continuous at $(0,0)$ (for example we could use the sequential characterization of limits with $\lim_{k\to\infty} f(\frac{1}{k}, \frac{1}{k}) = \lim_{k\to\infty} 1 = 1$ and $\lim_{k\to\infty} f(\frac{1}{k}, 0) = \lim_{k\to\infty} 0 = 0$). However,

$$g(x) = f(x,0) = 0$$

for all $x \in (-1,1)$ so g is continuous on $(-1,1)$ since g is constant. Similarly

$$h(y) = f(0,y) = 0$$

for all $y \in (-1,1)$ so h is continuous on $(-1,1)$ since h is constant. □

(4) Find an example of a continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$ and a closed set $F \subset \mathbb{R}^n$ so that $f(F)$ not a closed set.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{x}{\sqrt{1+x^2}}$ A straightforward computation shows that the function $g : (-1, 1) \to \mathbb{R}$ defined by $g(x) = \frac{x}{\sqrt{1-x^2}}$ satisfies

$$f(g(x)) = x \quad \text{for all } x \in (-1, 1)$$

and

$$g(f(x)) = x \quad \text{for all } x \in \mathbb{R}.$$

f is invertible and $f^{-1} = g$. We can conclude from this that $f(\mathbb{R}) = (-1, 1)$. Then the image of the closed set $\mathbb{R}$ is $(-1, 1)$ which is not closed. □

Remark. Note that in constructing your example, the closed set F must be unbounded. This is because if F were both closed and bounded, the Heine-Borel Theorem would tell us that F is compact, and then $f(F)$ would be compact, and hence closed and bounded, by Theorem 9.29. Therefore if we are to have $f(F)$ not closed with f continuous and F closed, it must be the case that F is not bounded.