Math 421, Homework #9 Solutions

(b) Prove that open balls in \( \mathbb{R}^n \) are connected, i.e. given \( a \in \mathbb{R}^n \) and \( r > 0 \) prove that \( B_r(a) \) is connected.

Proof. Applying part (a) it suffices to show that an open ball is always path connected. Let \( a \in \mathbb{R}^n \), let \( r > 0 \), and let \( x, y \in B_r(a) \).

Define \( \gamma : [0,1] \to \mathbb{R}^n \) by

\[
\gamma(t) = (1-t)x + ty.
\]

Then \( \gamma \) is a continuous function since each component function is a first-order polynomial in \( t \). Moreover \( \gamma(0) = x \) and \( \gamma(1) = y \). We claim that \( \gamma(t) \in B_r(a) \) for all \( t \in [0,1] \). Indeed for \( t = 0 \) or \( t = 1 \) we have that \( \gamma(t) \in B_r(a) \) by assumption. Moreover, for \( t \in (0,1) \) we have that

\[
\|\gamma(t) - a\| = \|(1-t)x + ty - a\|
\]

\[
= \|(1-t)(x - a) + t(y - a)\|
\]

\[
\leq |1-t|\|x - a\| + |t|\|y - a\|
\]

\[
= |1-t|\|x - a\| + |t|\|y - a\|
\]

\[
< |1-t|r + |t|r
\]

\[
= (1-t)r + tr = r
\]

\[
x, y \in B_r(a), t \neq 0 \text{ and } t \neq 1
\]

We conclude that \( \gamma(t) \in B_r(a) \) for all \( t \in [0,1] \). Therefore \( B_r(a) \) is path-connected, and by part (a), \( B_r(a) \) is connected. \( \square \)
(c) Prove that $\mathbb{R}^n$ is connected.

Proof. Arguing as in part (b), given $x, y \in \mathbb{R}^n$, the function $\gamma : [0, 1] \to \mathbb{R}^n$ defined by

$$\gamma(t) = (1 - t)x + ty$$

is a continuous function with $\gamma(0) = x$, $\gamma(1) = y$ and $\gamma(t) \in \mathbb{R}^n$ for all $t \in [0, 1]$. Therefore $\mathbb{R}^n$ is path connected, and hence connected by part (a). □

Alternate proof. We have that $\mathbb{R}^n = \bigcup_{k \in \mathbb{N}} B_k(0)$ and that $\bigcap_{k \in \mathbb{N}} B_k(0) = B_1(0) \neq \emptyset$. Therefore problem 2(b) from Homework #5 tells us that $\mathbb{R}^n$ is connected since each of the sets $B_k(0)$ is connected. □

(d) Prove that only subsets of $\mathbb{R}^n$ which are both open and closed are $\mathbb{R}^n$ and $\emptyset$.

Proof. Assume that $U \subset \mathbb{R}^n$ is open and closed, and that $U \neq \mathbb{R}^n$ and $U \neq \emptyset$. We claim that $U$ and $V := U^c$ separate $\mathbb{R}^n$. Indeed $U$ is open (and hence relatively open in $\mathbb{R}^n$) and nonempty by assumption. Since $U$ is also assumed to be closed, $V = U^c$ is also open. Moreover, since we assume that $U \neq \mathbb{R}^n$ it must by that $V = \mathbb{R}^n \setminus U$ is nonempty. Finally we have that

$$U \cap V = U \cap U^c = \emptyset$$

and that

$$U \cup V = U \cup U^c = \mathbb{R}^n.$$

Therefore $U$ and $V$ separate $\mathbb{R}^n$ and therefore $\mathbb{R}^n$ is not connected in contradiction part to (c). This contradiction lets us conclude that any subset $U$ of $\mathbb{R}^n$ that is both open and closed must satisfy either $U = \mathbb{R}^n$ or $U = \emptyset$. □
(2) Consider a function $f : E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ and assume that $f$ is continuous at some point $a \in E$ and that $f(a) \neq 0$. Prove that there is an $r > 0$ so that for all $x \in E$ with $\|x - a\| < r$, $f(x) \neq 0$.

**Proof.** Since $f(a) \neq 0$ and $f$ is assumed to be continuous at $a$ we can find an $r > 0$ so that $\|f(x) - f(a)\| < \frac{1}{2} \|f(a)\| (\neq 0)$ for $x \in E$ with $\|x - a\| < r$.

Using the triangle inequality, it follows that for $x \in E$ with $\|x - a\| < r$

$$\|f(x)\| - \|f(a)\| \leq \|f(x) - f(a)\| < \frac{1}{2} \|f(a)\|$$

or equivalently that

$$\frac{1}{2} \|f(a)\| < \|f(x)\| - \|f(a)\| < \frac{1}{2} \|f(a)\|$$

and hence

$$\|f(x)\| > \frac{1}{2} \|f(a)\| \neq 0 \quad \text{for } x \in E \text{ with } \|x - a\| < r.$$ 

Therefore $f(x) \neq 0$ for $x \in E$ with $\|x - a\| < r$. □

**Alternate proof.** Assume not. Then for every $r > 0$ there is an $x_r \in E$ satisfying $\|x_r - a\| < r$ and $f(x_r) = 0$. In particular, for each $k \in \mathbb{N}$ there is an $x_k \in E$ satisfying $\|x_k - a\| < \frac{1}{k}$ and $f(x_k) = 0$.

But then we can use the squeeze theorem with $\|x_k - a\| < \frac{1}{k}$ to conclude that $x_k \rightarrow a$ so $x_k$ is a sequence in $E$ converging to $a$. Consequently, the sequential characterization of continuity and the assumption that $f$ is continuous at $a$ let’s us conclude that

$$\lim_{k \rightarrow \infty} f(x_k) = f(a).$$

However, since $f(x_k) = 0$ for all $k \in \mathbb{N}$, this leads to the contradiction

$$0 \neq f(a) = \lim_{k \rightarrow \infty} f(x_k) = \lim_{k \rightarrow \infty} 0 = 0.$$

This contradiction allows us to conclude the original statement is true. □
Let $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$ be open intervals and let $(a, b) \in I \times J$. Given a function $f : I \times J \to \mathbb{R}$ define functions $g : I \to \mathbb{R}$ and $h : J \to \mathbb{R}$ by

$$g(x) = f(x, b) \quad \text{and} \quad h(x) = f(a, x).$$

(a) Assume that $f$ is continuous at $(a, b)$. Prove that $g$ is continuous at $a$ and that $h$ is continuous at $b$.

**Proof.** Let $\varepsilon > 0$ Since $f$ is assumed to be continuous at $(a, b)$, there exists a $\delta > 0$ so that

$$|f(x, y) - f(a, b)| < \varepsilon \quad \text{for} \quad (x, y) \in I \times J \quad \text{with} \quad \|(x, y) - (a, b)\| < \delta.$$ 

Then if $x \in I$ and $|x - a| < \delta$, we have $\|(x, b) - (a, b)\| = \sqrt{(x - a)^2} = |x - a| < \delta$ so we can use the definition of $g$ to conclude that

$$|g(x) - g(a)| = |f(x, b) - f(a, b)| < \varepsilon.$$ 

Therefore $g$ is continuous at $a$. Similarly, we have that if $y \in J$ and $|y - b| < \delta$, then $\|(a, y) - (a, b)\| = |y - b| < \delta$ so

$$|h(y) - h(b)| = |f(a, y) - f(a, b)| < \varepsilon,$$

and hence $h$ is continuous at $b$. \qed

(b) Show that the converse of part (a) is not true, i.e. find an example of a function $f : I \times J \to \mathbb{R}$ which is not continuous at $(a, b)$ but where $g$ is continuous at $a$ and $h$ is continuous at $b$ (with $g$ and $h$ defined as above).

**Example.** Let $f : (-1, 1) \times (-1, 1) \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} 
0 & xy = 0 \\
1 & xy \neq 0
\end{cases}$$

and let $(a, b) = (0, 0)$. Then it’s easy to show that $f$ is not continuous at $(0, 0)$ (for example we could use the sequential characterization of limits with $\lim_{k \to \infty} f\left(\frac{1}{k}, \frac{1}{k}\right) = \lim_{k \to \infty} 1 = 1$ and $\lim_{k \to \infty} f\left(\frac{1}{k}, 0\right) = \lim_{k \to \infty} 0 = 0$). However,

$$g(x) = f(x, 0) = 0$$

for all $x \in (-1, 1)$ so $g$ is continuous on $(-1, 1)$ since $g$ is constant. Similarly

$$h(y) = f(0, y) = 0$$

for all $y \in (-1, 1)$ so $h$ is continuous on $(-1, 1)$ since $h$ is constant. \qed
(4) Find an example of a continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$ and a closed set $F \subset \mathbb{R}^n$ so that $f(F)$ not a closed set.

**Example.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{x}{\sqrt{1+x^2}}$. A straightforward computation shows that the function $g : (-1,1) \to \mathbb{R}$ defined by $g(x) = \frac{x}{\sqrt{1-x^2}}$ satisfies

$$f(g(x)) = x \quad \text{for all } x \in (-1,1)$$

and

$$g(f(x)) = x \quad \text{for all } x \in \mathbb{R}.$$ 

$f$ is invertible and $f^{-1} = g$. We can conclude from this that $f(\mathbb{R}) = (-1,1)$. Then the image of the closed set $\mathbb{R}$ is $(-1,1)$ which is not closed. \qed

**Remark.** Note that in constructing your example, the closed set $F$ must be unbounded. This is because if $F$ were both closed and bounded, the Heine-Borel Theorem would tell us that $F$ is compact, and then $f(F)$ would be compact, and hence closed and bounded, by Theorem 9.29. Therefore if we are to have $f(F)$ not closed with $f$ continuous and $F$ closed, it must be the case that $F$ is not bounded.