Math 421, Homework #9
Due: Wednesday, April 14

Homework assignments should be submitted during lecture. Please consult the homework rules and guidelines before completing the assignment. In particular:

- Write in compete sentences, organized into paragraphs.
- Leave plenty of room in between problems.
- Write only on the front side of each sheet of paper.
- Staple!
- Write on your assignment the names of any persons or sources consulted during its completion (other than the course text or instructor).

(1) (a) A set $E \subset \mathbb{R}^n$ is said to be *path connected* if for any pair of points $\mathbf{x} \in E$ and $\mathbf{y} \in E$ there exists a continuous function $\boldsymbol{\gamma} : [0,1] \to \mathbb{R}^n$ satisfying $\boldsymbol{\gamma}(0) = \mathbf{x}$, $\boldsymbol{\gamma}(1) = \mathbf{y}$, and $\boldsymbol{\gamma}(t) \in E$ for all $t \in [0,1]$. Let $E \subset \mathbb{R}^n$ and assume that E is path connected. Prove that E is connected.

(b) Prove that open balls in $\mathbb{R}^n$ are connected, i.e. given $\mathbf{a} \in \mathbb{R}^n$ and $r > 0$ prove that $B_r(\mathbf{a})$ is connected.

(c) Prove that $\mathbb{R}^n$ is connected.

(d) Prove that only subsets of $\mathbb{R}^n$ which are both open and closed are $\mathbb{R}^n$ and $\emptyset$.

(2) Consider a function $\mathbf{f} : E \subset \mathbb{R}^n \to \mathbb{R}^m$ and assume that $\mathbf{f}$ is continuous at some point $\mathbf{a} \in E$ and that $\mathbf{f}(\mathbf{a}) \neq \mathbf{0}$. Prove that there is an $r > 0$ so that for all $\mathbf{x} \in E$ with $\|\mathbf{x} - \mathbf{a}\| < r$, $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$.

(3) Let $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$ be open intervals and let $(a,b) \in I \times J$. Given a function $f : I \times J \to \mathbb{R}$ define functions $g : I \to \mathbb{R}$ and $h : J \to \mathbb{R}$ by

$$g(x) = f(x,b) \text{ and } h(x) = f(a,x).$$

(a) Assume that f is continuous at (a,b). Prove that g is continuous at a and that h is continuous at b.

(b) Show that the converse of part (a) is not true, i.e. find an example of a function $f : I \times J \to \mathbb{R}$ which is not continuous at (a,b) but where g is continuous at a and h is continuous at b (with g and h defined as above).

(4) Find an example of a continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$ and a closed set $F \subset \mathbb{R}^n$ so that $f(F)$ not a closed set.