Math 421, Homework #8 Solutions

(1) Find an example of a function $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ for which $\lim_{(x,y)\to\mathbf{0}} f(x, y)$ exists, but the iterated limits $\lim_{x\to 0} \lim_{y\to 0} f(x, y)$ and $\lim_{y\to 0} \lim_{x\to 0} f(x, y)$ do not exist.

Answer. Note that many correct answers are possible here. One way to construct an example is by considering a function $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$g \text{ is bounded,} \tag{1}$$
$$\lim_{x\to 0} g(x) \text{ does not exist, and} \tag{2}$$
$$g(x) \neq 0 \text{ for all } x \in \mathbb{R} \setminus \{0\}. \tag{3}$$

(We will address the existence of such a function below.)

Given a g satisfying (1)–(3), we define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \left(x^2 + y^2\right) g(x)g(y).$$

We first claim that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$. Indeed, since g is assumed to be bounded, there exists an $M > 0$ so that $|g(x)| \le M$ for all $x \in \mathbb{R}$. Therefore

$$|f(x, y)| = \left(x^2 + y^2\right) |g(x)| \, |g(y)| \le \left(x^2 + y^2\right) M^2 = M^2 \left\|(x, y)\right\|^2 .$$

Given $\varepsilon > 0$, we choose $\delta = \sqrt{\varepsilon}/M$. Then for $|(x, y)| < \delta$ we have that

$$|f(x, y) - 0| \le M^2 \left\|(x, y)\right\|^2 < M^2 \left(\sqrt{\varepsilon}/M\right)^2 = \varepsilon.$$

We conclude that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$ as claimed.

We next claim that for fixed $y_0 \neq 0$ that $\lim_{x\to 0} f(x, y_0)$ does not exist. Arguing by contradiction, assume that $\lim_{x\to 0} f(x, y_0) = L$. Then, since

$$g(x) = \frac{1}{\left(x^2 + y_0^2\right) g(y_0)} f(x, y_0)$$

we could use Theorem 9.15 with the assumptions that $y_0 \neq 0$ and $g(y_0) \neq 0$ to conclude that

$$\begin{aligned} \lim_{x\to 0} g(x) &= \lim_{x\to 0} \left(\frac{1}{\left(x^2 + y_0^2\right) g(y_0)} f(x, y_0) \right) \\ &= \lim_{x\to 0} \frac{1}{\left(x^2 + y_0^2\right) g(y_0)} \lim_{x\to 0} f(x, y_0) \\ &= \frac{L}{y_0^2 g(y_0)} \end{aligned}$$

in contradiction to the assumption that $\lim_{x\to 0} g(x)$ does not exist. We conclude that $\lim_{x\to 0} f(x, y_0)$ does not exist for any fixed $y_0 \neq 0$, and that consequently, the iterated limit $\lim_{y\to 0} \lim_{x\to 0} f(x, y)$ does not exist either. Since f is symmetric in x and y, an identical argument shows that similarly $\lim_{y\to 0} f(x_0, y)$ does not exist for any fixed $x_0 \neq 0$, and hence that $\lim_{x\to 0} \lim_{y\to 0} f(x, y)$ does not exist.

Finally, we address the existence of a function g satisfying (1)–(3). The following examples are easily verified to satisfy (1)–(3):

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$$

and

$$g(x) = \begin{cases} 2 + \sin \frac{1}{x} & x \neq 0 \\ 2 & x = 0. \end{cases}$$

□

(2) Let $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$ be open intervals, let $(a, b) \in I \times J$, and consider functions $g : J \setminus \{b\} \to \mathbb{R}$ and $f : (I \setminus \{a\}) \times (J \setminus \{b\}) \to \mathbb{R}$. We say that

$$\lim_{x \to a} f(x, y) = g(y) \text{ \textit{uniformly} for } y \in J \setminus \{b\}$$

if for any $\varepsilon > 0$ there is a $\delta > 0$ so that for all $x \in I \setminus \{a\}$ with $0 < |x - a| < \delta$, and all $y \in J \setminus \{b\}$

$$|f(x, y) - g(y)| < \varepsilon.$$

With I, J, a, b, f, and g as above, assume that $\lim_{x \to a} f(x, y) = g(y)$ uniformly for $y \in J \setminus \{b\}$, and that $\lim_{y \to b} g(y) = L$. Prove that

$$\lim_{(x,y) \to (a,b)} f(x, y) = L.$$

Proof. Let $\varepsilon > 0$. By the assumption that $\lim_{x \to a} f(x, y) = g(y)$ uniformly for $y \in J \setminus \{b\}$ we can choose a $\delta_1 > 0$ so that for x satisfying $0 < |x - a| < \delta_1$ and all $y \in J \setminus \{b\}$

$$|f(x, y) - g(y)| < \varepsilon/2.$$

Similarly, since $\lim_{y \to b} g(y) = L$, we can find a $\delta_2 > 0$ so that for y satisfying $0 < |y - b| < \delta_2$,

$$|g(y) - L| < \varepsilon/2.$$

Define $\delta = \min\{\delta_1, \delta_2\}$. Then, if $\|(x, y) - (a, b)\| < \delta$, we have that

$$|x - a| = \sqrt{(x - a)^2} \leq \sqrt{(x - a)^2 + (y - b)^2} = \|(x - a, y - b)\| = \|(x, y) - (a, b)\| < \delta \leq \delta_1$$

and similarly

$$|y - b| = \sqrt{(y - b)^2} \leq \sqrt{(x - a)^2 + (y - b)^2} = \|(x, y) - (a, b)\| < \delta \leq \delta_2.$$

Therefore, for $(x, y) \in (I \setminus \{a\}) \times (J \setminus \{b\})$ satisfying $\|(x, y) - (a, b)\| < \delta$, we can combine the above to conclude that

$$\begin{aligned} |f(x, y) - L| &= |(f(x, y) - g(y)) + (g(y) - L)| \\ &\leq |f(x, y) - g(y)| + |g(y) - L| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

We conclude that $\lim_{(x,y) \to (a,b)} f(x, y) = L$.

$\square$

(3) (9.4.6) Prove that

$$f(x,y) = \begin{cases} e^{-1/|x-y|} & x \neq y \\ 0 & x = y \end{cases}$$

is continuous on $\mathbb{R}^2$.

Proof. Since for any (x, y), $(a, b) \in \mathbb{R}^2$ we have (arguing as in problem 2) that

$$|x - a| \leq \|(x, y) - (a, b)\| \tag{4}$$

and

$$|y - b| \leq \|(x, y) - (a, b)\| \tag{5}$$

it is straightforward to show that the functions $p_1(x, y) = x$ and $p_2(x, y) = y$ are continuous (indeed uniformly continuous) at every point in $\mathbb{R}^2$ (in either case, given $\varepsilon > 0$ choose $\delta = \varepsilon$). Moreover, since sums/products/compositions of continuous functions are continuous, and reciprocals of continuous functions are continuous where the function is nonzero, we can conclude that $e^{-1/|x-y|}$ is continuous wherever it is defined, so $f(x, y)$ is continuous at any point in $\mathbb{R}^2$ with $x \neq y$.

It remains to show that $f(x, y)$ is continuous at points where $x = y$. Consider a point $(a, a) \in \mathbb{R}^2$, and let $\varepsilon > 0$. Choose $\delta = \frac{1}{2}(-\log c_\varepsilon)^{-1}$ where $c_\varepsilon := \min\{1/2, \varepsilon\}$. Note that since $c_\varepsilon \leq 1/2 < 1$, it follows that $\log c_\varepsilon < 0$ so $\delta > 0$. Then, if $\|(x, y) - (a, a)\| < \delta$ it follows from (4) and (5) that

$$\begin{aligned} |x - y| &= |(x - a) + (a - y)| \\ &\leq |x - a| + |a - y| \\ &= |x - a| + |y - a| \\ &\leq \|(x, y) - (a, a)\| + \|(x, y) - (a, a)\| \\ &= 2\,\|(x, y) - (a, a)\| + \|(x, y) - (a, a)\| \\ &< 2\delta. \end{aligned}$$

We can conclude that for $\|(x, y) - (a, a)\| < \delta$ with $x \neq y$ that $1/|x - y| > 1/(2\delta)$ and thus that $-1/|x - y| < -1/(2\delta)$. Since e^t is a strictly increasing function, we can conclude that for $x \neq y$ satisfying $\|(x, y) - (a, a)\| < \delta$,

$$|f(x, y) - 0| = e^{-1/|x-y|} < e^{-1/(2\delta)} = e^{\log c_\varepsilon} = c_\varepsilon \leq \varepsilon.$$

Meanwhile, if $x = y$ we have that

$$|f(x, y) - 0| = 0 < \varepsilon.$$

Therefore, for $(x, y) \in \mathbb{R}^2$ satisfying $|(x, y) - (a, a)| < \delta$, we have that

$$|f(x, y) - 0| < \varepsilon,$$

so we can conclude that f is continuous at $(a, a) \in \mathbb{R}^2$. Combining this with the observations of the first paragraph, we can conclude that f is continuous at every point in $\mathbb{R}^2$. $\square$

(4) Let $E \subset \mathbb{R}^n$ be a bounded set, and assume that $\mathbf{f} : E \to \mathbb{R}^m$ is uniformly continuous on E. Show that $\mathbf{f}$ is a bounded function, i.e. show that there exists an $M > 0$ so that for every $\mathbf{x} \in E$, $\|\mathbf{f}(\mathbf{x})\| \leq M$.

Proof. Since $\mathbf{f}$ is uniformly continuous on E, we can find a $\delta > 0$ so that if $\mathbf{x}$, $\mathbf{y} \in E$ satisfy $\|\mathbf{x} - \mathbf{y}\| < \delta$,

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < 1.$$

Assume that $\mathbf{f}$ is not bounded. Then for every $M > 0$ there exists an $\mathbf{x} \in E$ with $\|\mathbf{f}(\mathbf{x})\| > M$. In particular, we can construct a sequence, by choosing any $\mathbf{x}_1 \in E$, and then inductively choose $\mathbf{x}_k \in E$ to satisfy

$$\|\mathbf{f}(\mathbf{x}_k)\| \geq 1 + \|\mathbf{f}(\mathbf{x}_{k-1})\|.$$

A straightforward induction argument then shows that

$$\|\mathbf{f}(\mathbf{x}_{k_1})\| - \|\mathbf{f}(\mathbf{x}_{k_2})\| \geq k_1 - k_2 \quad \text{for all } k_1,\, k_2 \in \mathbb{N} \text{ with } k_1 > k_2. \tag{6}$$

Since E is bounded and $\mathbf{x}_k \in E$ for all $k \in \mathbb{N}$, it follows that $\{\mathbf{x}_k\}$ is a bounded sequence. By the Bolzano-Weierstrass theorem, we can find a convergent subsequence $\left\{\mathbf{x}_{k_j}\right\}_{j \in \mathbb{N}}$ (the limit of which might not be in E). Since $\mathbf{x}_{k_j}$ is convergent, it is a Cauchy sequence. In particular, there exists an $N \in \mathbb{N}$ so that for $j > \ell \geq N$,

$$\left\|\mathbf{x}_{k_j} - \mathbf{x}_{k_\ell}\right\| < \delta$$

with δ as chosen in the first paragraph. But then it follows from the first paragraph that

$$\left\|\mathbf{f}(\mathbf{x}_{k_j}) - \mathbf{f}(\mathbf{x}_{k_\ell})\right\| < 1.$$

We then arrive at the contradiction $1 > 1$ arguing as follows:

$$\begin{aligned}
1 &> \left\|\mathbf{f}(\mathbf{x}_{k_j}) - \mathbf{f}(\mathbf{x}_{k_\ell})\right\| \\
&\geq \left|\|\mathbf{f}(\mathbf{x}_{k_j})\| - \|\mathbf{f}(\mathbf{x}_{k_\ell})\|\right| && \text{by the triangle inequality} \\
&\geq \|\mathbf{f}(\mathbf{x}_{k_j})\| - \|\mathbf{f}(\mathbf{x}_{k_\ell})\| \\
&\geq k_j - k_\ell && \text{by (6) and } j > \ell \\
&\geq 1
\end{aligned}$$

where in the last two inequalities we've used that k_j is a strictly increasing sequence of natural numbers (from the definition of subsequence). This contradiction let's us conclude that $\mathbf{f}$ is bounded. □

(5) (cf. 9.4.8) Let $E \subset \mathbb{R}^n$ and let $\mathbf{f} : E \to \mathbb{R}^m$ be uniformly continuous on E. Prove that $\mathbf{f}$ can be extended to a continuous function on the closure $\bar{E}$ of E, i.e. prove that there exists a continuous function $\mathbf{g} : \bar{E} \to \mathbb{R}^m$ satisfying $\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in E$.

Proof. Let $\mathbf{a} \in \bar{E} \setminus E$. The assumption that $\mathbf{a} \in \bar{E}$ implies that every set of the form $B_\varepsilon(\mathbf{a}) \cap E$ is nonempty, and since $a \notin E$ by assumption, it follows that for every $\varepsilon > 0$, $B_\varepsilon(\mathbf{a}) \cap E$ contains points that are not equal to $\mathbf{a}$. Applying this statement to the sequence $\varepsilon_n = \frac{1}{n}$, it follows that exist sequences E which converge to $\mathbf{a}$.

Let $\{\mathbf{x}_k\}$ be a sequence in E converging to $\mathbf{a}$. We claim that $\mathbf{f}(\mathbf{x}_k)$ is a convergent sequence. To show this, it suffices to show that $\mathbf{f}(\mathbf{x}_k)$ is a Cauchy sequence. Let $\varepsilon > 0$. Using the assumption that $\mathbf{f}$ is uniformly continuous on E, we can find a $\delta > 0$ so that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < \varepsilon \quad \text{for all } \mathbf{x}, \mathbf{y} \in E \text{ with } \|\mathbf{x} - \mathbf{y}\| < \delta.$$

Then, since $\{\mathbf{x}_k\}$ is convergent, it is a Cauchy sequence, so there exists an $N \in \mathbb{N}$ so that

$$\|\mathbf{x}_k - \mathbf{x}_j\| < \delta \quad \text{for all } k, j \geq N.$$

Combining these statements, we find that for k, $j \geq N$, that

$$\|\mathbf{f}(\mathbf{x}_k) - \mathbf{f}(\mathbf{x}_j)\| < \varepsilon.$$

We conclude that $\mathbf{f}(\mathbf{x}_k)$ is a Cauchy sequence and therefore a convergent sequence.

We next show that $\lim_{k\to\infty} \mathbf{f}(\mathbf{x}_k)$ is independent of the choice of sequence $\mathbf{x}_k \to \mathbf{a}$. Consider two sequences $\{\mathbf{x}_k\}$ and $\{\mathbf{y}_k\}$ in E, and assume both sequences converge to $\mathbf{a}$. From the previous paragraph, both of the sequences $\{\mathbf{f}(\mathbf{x}_k)\}$ and $\{\mathbf{f}(\mathbf{y}_k)\}$ are convergent. Let $\mathbf{L}_1 = \lim_{k\to\infty} \mathbf{f}(\mathbf{x}_k)$ and let $\mathbf{L}_2 = \lim_{k\to\infty} \mathbf{f}(\mathbf{y}_k)$. We seek to show that $\mathbf{L}_1 = \mathbf{L}_2$. Let $\varepsilon > 0$. The assumption that $\mathbf{f}$ is uniformly continuous on E again allows us to find a $\delta > 0$ so that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < \varepsilon/2 \quad \text{for all } \mathbf{x}, \mathbf{y} \in E \text{ with } \|\mathbf{x} - \mathbf{y}\| < \delta.$$

Then, since $\mathbf{x}_k$ and $\mathbf{y}_k$ both converge to $\mathbf{a}$, we can find an $N \in \mathbb{N}$ so that $\|\mathbf{x}_k - \mathbf{a}\| < \delta/2$ and $\|\mathbf{y}_k - \mathbf{a}\| < \delta/2$ for $k \geq N$, and consequently

$$\|\mathbf{x}_k - \mathbf{y}_k\| \leq \|\mathbf{x}_k - \mathbf{a}\| + \|\mathbf{a} - \mathbf{y}_k\| < \delta/2 + \delta/2 = \delta \quad \text{for } k \geq N.$$

Combining the above, we find that

$$\|\mathbf{f}(\mathbf{x}_k) - \mathbf{f}(\mathbf{y}_k)\| < \varepsilon/2 \quad \text{for } k \geq N.$$

Letting $k \to \infty$ above and using Theorem 2.17 and the comments following Theorem 9.4, we find that

$$\|\mathbf{L}_1 - \mathbf{L}_2\| = \lim_{k\to\infty} \|\mathbf{f}(\mathbf{x}_k) - \mathbf{f}(\mathbf{y}_k)\| \leq \varepsilon/2 < \varepsilon.$$

Thus $\|\mathbf{L}_1 - \mathbf{L}_2\| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\|\mathbf{L}_1 - \mathbf{L}_2\| = 0$ and hence that $\mathbf{L}_1 = \mathbf{L}_2$.

We now define $\mathbf{g} : \bar{E} \to \mathbb{R}^m$ as follows. If $\mathbf{a} \in E$ we define $\mathbf{g}(\mathbf{a}) = \mathbf{f}(\mathbf{a})$. If $\mathbf{a} \in \bar{E} \setminus E$, choose a sequence $\mathbf{x}_k \in E$ converging to $\mathbf{a}$ (which can be done by the comments in the first paragraph). Define $\mathbf{g}(\mathbf{a})$ by

$$\mathbf{g}(\mathbf{a}) := \lim_{k\to\infty} \mathbf{f}(\mathbf{x}_k).$$

This limit exists by the discussion in the second paragraph, and is independent of the choice of sequence $\mathbf{x}_k \to \mathbf{a}$ by the discussion in the third paragraph.

It remains to show that $\mathbf{g}$ defined in this way is a continuous function. Let $\varepsilon > 0$. We need to show that there is a $\delta > 0$ so that[1]

$$\|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})\| < \varepsilon \quad \text{for all } \mathbf{x}, \mathbf{y} \in \bar{E} \text{ with } \|\mathbf{x} - \mathbf{y}\| < \delta.$$

Using the uniform continuity of $\mathbf{f}$, we can choose a $\delta_1 > 0$ so that

$$\|\mathbf{f}(\mathbf{x}') - \mathbf{f}(\mathbf{y}')\| < \varepsilon/2 \quad \text{for all } \mathbf{x}', \mathbf{y}' \in E \text{ with } \|\mathbf{x}' - \mathbf{y}'\| < \delta_1. \tag{7}$$

[1] This will actually show that $\mathbf{g}$ is uniformly continuous on $\bar{E}$, which is slightly stronger than what you are asked to prove.

Define $\delta = \delta_1/2$, and assume that $\mathbf{x}$, $\mathbf{y} \in \bar{E}$ satisfy $\|\mathbf{x} - \mathbf{y}\| < \delta$. Choose sequences $\{\mathbf{x}_k\}$ and $\{\mathbf{y}_k\}$ in E satisfying $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}$ and $\lim_{k\to\infty} \mathbf{y}_k = \mathbf{y}$, and note that by definition of $\mathbf{g}$ we have that $\lim_{k\to\infty} \mathbf{f}(\mathbf{x}_k) = \mathbf{g}(\mathbf{x})$ and $\lim_{k\to\infty} \mathbf{f}(\mathbf{x}_k) = \mathbf{g}(\mathbf{x})$. Since $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}$ and $\lim_{k\to\infty} \mathbf{y}_k = \mathbf{y}$, we can find an $N \in \mathbb{N}$ so that

$$\|\mathbf{x}_k - \mathbf{x}\| < \delta/2 = \delta_1/4 \quad \text{and} \quad \|\mathbf{y}_k - \mathbf{y}\| < \delta/2 = \delta_1/4 \quad \text{for } k \geq N,$$

and consequently

$$\|\mathbf{x}_k - \mathbf{y}_k\| \leq \|\mathbf{x}_k - \mathbf{x}\| + \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{y}_k\| < \delta_1/4 + \delta_1/2 + \delta_1/4 = \delta_1 \quad \text{for } k \geq N.$$

Using this with (7) we can conclude that

$$\|\mathbf{f}(\mathbf{x}_k) - \mathbf{f}(\mathbf{y}_k)\| < \varepsilon/2 \quad \text{for } k \geq N.$$

Again using Theorem 2.17 and the comments following Theorem 9.4, we can take a limit here and conclude that

$$\|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})\| = \lim_{k\to\infty} \|\mathbf{f}(\mathbf{x}_k) - \mathbf{f}(\mathbf{y}_k)\| \leq \varepsilon/2 < \varepsilon.$$

To summarize, given $\varepsilon > 0$ we have shown how to choose $\delta > 0$ so that for $\mathbf{x}$, $\mathbf{y} \in \bar{E}$ with $\|\mathbf{x} - \mathbf{y}\| < \delta$, $\|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})\| < \varepsilon$. We conclude that $\mathbf{g}$ is uniformly continuous on $\bar{E}$. $\square$

Remark. In the part of the above proof where we show $\mathbf{g}$ to be continuous, one might be tempted to argue as follows: since, given any sequence $\{\mathbf{x}_k\}$ in E converging to $\mathbf{x} \in \bar{E}$, $\lim_{k\to\infty} \mathbf{g}(\mathbf{x}_k) = \lim_{k\to\infty} \mathbf{f}(\mathbf{x}_k) = \mathbf{g}(\mathbf{x})$, the sequential characterization of continuity allows one to conclude that $\mathbf{g}$ is continuous at $\mathbf{x} \in \bar{E}$. The problem with this argument is that it only considers sequences in E. In order to use the sequential characterization of continuity to show that $\mathbf{g}$ is continuous on $\bar{E}$, we have to consider sequences where some (or all) of the terms might be in $\bar{E} \setminus E$; that is, we can say $\mathbf{g}$ is continuous at $\mathbf{x} \in \bar{E}$ if and only if for every sequence $\{\mathbf{x}_k\}$ in $\bar{E}$ which converges to $\mathbf{x}$, $\lim_{k\to\infty} \mathbf{g}(\mathbf{x}_k) = \mathbf{g}(\mathbf{x})$. At the point in the problem where we are trying to show $\mathbf{g}$ is continuous, we only know $\lim_{k\to\infty} \mathbf{g}(\mathbf{x}_k) = \mathbf{g}(\mathbf{x})$ to be true if the each term in the sequence $\mathbf{x}_k$ is assumed to be in E.

The reason that the additional work we did here to prove that $\mathbf{g}$ is continuous is not necessary in the proof of Theorem 3.40 (where the special case $E = (a, b)$ and $\bar{E} = [a, b]$ is addressed) is because assuming E is an interval simplifies things significantly. Indeed any sequence $x_k \in \bar{E} \setminus \{b\} = [a, b)$ converging to b will satisfy $x_k \in E = (a, b)$ for sufficiently large values of k (and similarly if we replace "b" by "a" in this statement).

If were instead we consider the case $E = \mathbb{Q}$ and $\bar{E} = \mathbb{R}$, it should be clear why it takes more work to prove that the extended function g is continuous. To use the sequential characterization of continuity to show a function $g : \bar{E} \to \mathbb{R}$ is continuous at, say, $\sqrt{2}$, it is not sufficient to consider only sequences $x_k \to \sqrt{2}$ with $x_k \in \mathbb{Q}$. One must consider all possible sequences of real numbers $x_k \to \sqrt{2}$ where each x_k may be rational or irrational.