Math 421, Homework #8
Due: Wednesday, April 7

Homework assignments should be submitted during lecture. Please consult the homework rules and guidelines before completing the assignment. In particular:

- Write in compete sentences, organized into paragraphs.
- Leave plenty of room in between problems.
- Write only on the front side of each sheet of paper.
- Staple!
- Write on your assignment the names of any persons or sources consulted during its completion (other than the course text or instructor).

(1) Find an example of a function $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ for which $\lim_{(x,y)\to\mathbf{0}} f(x,y)$ exists, but the iterated limits $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ and $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ do not exist.
(This shows that the explicit assumption that iterated limits $\lim_{x\to a}\lim_{y\to b} f(x,y)$ and $\lim_{y\to b}\lim_{x\to a} f(x,y)$ exist can't be dropped from Remark 9.22.)

(2) Let $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$ be open intervals, let $(a,b) \in I \times J$, and consider functions $g : J \setminus \{b\} \to \mathbb{R}$ and $f : (I \setminus \{a\}) \times (J \setminus \{b\}) \to \mathbb{R}$. We say that

$$\lim_{x\to a} f(x,y) = g(y) \textit{ uniformly} \text{ for } y \in J \setminus \{b\}$$

if for any $\varepsilon > 0$ there is a $\delta > 0$ so that for all $x \in I \setminus \{a\}$ with $0 < |x-a| < \delta$, and all $y \in J \setminus \{b\}$

$$|f(x,y) - g(y)| < \varepsilon.$$ [1]

With I, J, a, b, f, and g as above, assume that $\lim_{x\to a} f(x,y) = g(y)$ uniformly for $y \in J \setminus \{b\}$, and that $\lim_{y\to b} g(y) = L$. Prove that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

(3) (9.4.6) Prove that

$$f(x,y) = \begin{cases} e^{-1/|x-y|} & x \neq y \\ 0 & x = y \end{cases}$$

is continuous on $\mathbb{R}^2$.

(4) Let $E \subset \mathbb{R}^n$ be a bounded set, and assume that $\mathbf{f} : E \to \mathbb{R}^m$ is uniformly continuous on E. Show that $\mathbf{f}$ is a bounded function, i.e. show that there exists an $M > 0$ so that for every $\mathbf{x} \in E$, $\|\mathbf{f}(\mathbf{x})\| \leq M$.

(5) (cf. 9.4.8) Let $E \subset \mathbb{R}^n$ and let $\mathbf{f} : E \to \mathbb{R}^m$ be uniformly continuous on E. Prove that $\mathbf{f}$ can be extended to a continuous function on the closure $\bar{E}$ of E, i.e. prove that there exists a continuous function $\mathbf{g} : \bar{E} \to \mathbb{R}^m$ satisfying $\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in E$.

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[1] Make sure you understand how

$$\lim_{x\to a} f(x,y) = g(y) \text{ uniformly for } y \in J \setminus \{b\}$$

is a different (and stronger) assumption than

$$\lim_{x\to a} f(x,y) = g(y) \text{ for all } y \in J \setminus \{b\}.$$