Math 421, Homework #7 Solutions

(1) Let $\{\mathbf{x}_k\}$ and $\{\mathbf{y}_k\}$ be convergent sequences in $\mathbb{R}^n$, and assume that $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{L}$ and that $\lim_{k\to\infty} \mathbf{y}_k = \mathbf{M}$. Prove directly from definition 9.1 (i.e. don't use Theorem 9.2) that:

(a) $\lim_{k\to\infty} \mathbf{x}_k + \mathbf{y}_k = \mathbf{L} + \mathbf{M}$.

Proof. Let $\varepsilon > 0$. The assumptions that $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{L}$ and $\lim_{k\to\infty} \mathbf{y}_k = \mathbf{M}$ allow us to find an $N \in \mathbb{N}$ so that for $k \geq N$,

$$\|\mathbf{x}_k - \mathbf{L}\| < \frac{\varepsilon}{2}$$

and

$$\|\mathbf{y}_k - \mathbf{M}\| < \frac{\varepsilon}{2}.$$

We can then us the triangle inequality to find for $k \geq N$ that

$$\begin{aligned} \|(\mathbf{x}_k + \mathbf{y}_k) - (\mathbf{L} + \mathbf{M})\| &= \|(\mathbf{x}_k - \mathbf{L}) + (\mathbf{y}_k - \mathbf{M})\| \\ &\leq \|\mathbf{x}_k - \mathbf{L}\| + \|\mathbf{y}_k - \mathbf{M}\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

We therefore conclude that $\lim_{k\to\infty} \mathbf{x}_k + \mathbf{y}_k = \mathbf{L} + \mathbf{M}$. $\square$

(b) $\lim_{k\to\infty} \mathbf{x}_k \cdot \mathbf{y}_k = \mathbf{L} \cdot \mathbf{M}$.

Proof. Let $\varepsilon > 0$.

We first consider the case that $\mathbf{M} = \mathbf{0}$. Then there exists an $N_1 \in \mathbb{N}$ so that

$$\|\mathbf{y}_k\| = \|\mathbf{y}_k - \mathbf{0}\| < \frac{\varepsilon}{1 + \|\mathbf{L}\|} \quad \text{for } k \geq N_1.$$

Similarly, there is an $N_2 \in \mathbb{N}$ so that

$$\|\mathbf{x}_k - \mathbf{L}\| < 1 \quad \text{for } k \geq N_2$$

and hence

$$\|\mathbf{x}_k\| = \|(\mathbf{x}_k - \mathbf{L}) + \mathbf{L}\| \leq \|\mathbf{x}_k - \mathbf{L}\| + \|\mathbf{L}\| < 1 + \|\mathbf{L}\| \quad \text{for } k \geq N_2.$$

Using the above with Cauchy-Schwartz inequality, we find for $k \geq N := \max\{N_1, N_2\}$

$$\begin{aligned} |\mathbf{x}_k \cdot \mathbf{y}_k - 0| &= |\mathbf{x}_k \cdot \mathbf{y}_k| \\ &\leq \|\mathbf{x}_k\| \, \|\mathbf{y}_k\| \\ &< (1 + \|\mathbf{L}\|) \frac{\varepsilon}{1 + \|\mathbf{L}\|} = \varepsilon. \end{aligned}$$

Therefore $\lim_{k\to\infty} \mathbf{x}_k \cdot \mathbf{y}_k = 0 = \mathbf{L} \cdot \mathbf{0} = \mathbf{L} \cdot \mathbf{M}$.

To deal with the case where $\mathbf{M} \neq \mathbf{0}$ we can argue as in the previous paragraph to find an $N_1 \in \mathbb{N}$ so that

$$\|\mathbf{x}_k\| < 1 + \|\mathbf{L}\| \quad \text{for } k \geq N_1.$$

We then use the assumption $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{L}$ to find an $N_2 \in \mathbb{N}$ so that

$$\|\mathbf{x}_k - \mathbf{L}\| < \frac{\varepsilon}{2\,\|\mathbf{M}\|} \quad \text{for } k \geq N_2,$$

and similarly, we can use the assumption $\lim_{k\to\infty} \mathbf{y}_k = \mathbf{M}$ to find an $N_3 \in \mathbb{N}$ so that

$$\|\mathbf{y}_k - \mathbf{M}\| < \frac{\varepsilon}{2\,(1 + \|\mathbf{L}\|)}.$$

Then we can use the Cauchy-Schwartz inequality to find for $k \geq N := \max\{N_1, N_2, N_3\}$ that

$$\begin{aligned} |\mathbf{x}_k \cdot \mathbf{y}_k - \mathbf{L} \cdot \mathbf{M}| &= |\mathbf{x}_k \cdot \mathbf{y}_k - \mathbf{x}_k \cdot \mathbf{M} + \mathbf{x}_k \cdot \mathbf{M} - \mathbf{L} \cdot \mathbf{M}| \\ &= |\mathbf{x}_k \cdot (\mathbf{y}_k - \mathbf{M}) + (\mathbf{x}_k - \mathbf{L}) \cdot \mathbf{M}| \\ &\leq |\mathbf{x}_k \cdot (\mathbf{y}_k - \mathbf{M})| + |(\mathbf{x}_k - \mathbf{L}) \cdot \mathbf{M}| \\ &\leq \|\mathbf{x}_k\| \, \|\mathbf{y}_k - \mathbf{M}\| + \|\mathbf{x}_k - \mathbf{L}\| \, \|\mathbf{M}\| \\ &< (1 + \|\mathbf{L}\|) \, \frac{\varepsilon}{2\,(1 + \|\mathbf{L}\|)} + \frac{\varepsilon}{2\,\|\mathbf{M}\|} \, \|\mathbf{M}\| = \varepsilon. \end{aligned}$$

Therefore $\lim_{k\to\infty} \mathbf{x}_k \cdot \mathbf{y}_k = \mathbf{L} \cdot \mathbf{M}$ as claimed. $\square$

(2) Prove directly from definition 9.10 (i.e. don't use the Heine-Borel Theorem or the Borel Covering Lemma) that if K_1 and K_2 are compact sets, then the union $K_1 \cup K_2$ is also compact.

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $K_1 \cup K_2$, i.e. we assume that all U_α are open sets, and that

$$K_1 \cup K_2 \subset \bigcup_{\alpha \in A} U_\alpha.$$

In order to show that $K_1 \cup K_2$ we need to find a finite subset A' of A so that $\{U_\alpha\}_{\alpha \in A'}$ is cover of $K_1 \cup K_2$. Since

$$K_1 \subset K_1 \cup K_2 \subset \bigcup_{\alpha \in A} U_\alpha,$$

$\{U_\alpha\}_{\alpha \in A}$ is an open cover of K_1. Since K_1 is assumed to be compact there is a finite subcover, i.e. there is a finite subset A_1 of A so that

$$K_1 \subset \bigcup_{\alpha \in A_1} U_\alpha.$$

Similarly, $\{U_\alpha\}_{\alpha \in A}$ is an open cover of K_2, and since K_2 is compact, we can find a finite subset A_2 of A so that

$$K_2 \subset \bigcup_{\alpha \in A_2} U_\alpha.$$

Combining the above, we have that

$$K_1 \cup K_2 \subset \left(\bigcup_{\alpha \in A_1} U_\alpha \right) \cup \left(\bigcup_{\alpha \in A_2} U_\alpha \right) = \bigcup_{\alpha \in A_1 \cup A_2} U_\alpha.$$

Since A_1 and A_2 are both finite, so is $A_1 \cup A_2$. Thus $\{U_\alpha\}_{\alpha \in A_1 \cup A_2}$ is a subcover of $\{U_\alpha\}_{\alpha \in A}$ with a finite number of sets in it. We conclude that $K_1 \cup K_2$ is compact. $\square$

(3) (9.2.4) Suppose that $K \subset \mathbb{R}^n$ is compact and that for every $\mathbf{x} \in K$ there is an $r(\mathbf{x}) > 0$ so that $B_{r(\mathbf{x})}(\mathbf{x}) \cap K = \{\mathbf{x}\}$. Prove that K is a finite set.

Proof using the definition of compactness. For each $\mathbf{x}$, we choose an $r(\mathbf{x}) > 0$ so that $B_{r(\mathbf{x})}(\mathbf{x}) \cap K = \{\mathbf{x}\}$. Then, since

$$K \subset \bigcup_{\mathbf{x} \in K} B_{r(\mathbf{x})}(\mathbf{x})$$

and each set $B_{r(\mathbf{x})}(\mathbf{x})$ is open, $\left\{B_{r(\mathbf{x})}(\mathbf{x})\right\}_{\mathbf{x} \in K}$ is an open cover for K. Since K is assumed to be compact, it follows that there exists a finite collection of point $\{\mathbf{x}_1, \dots, \mathbf{x}_j\} \subset K$ so that

$$K \subset \bigcup_{k=1}^{j} B_{r(\mathbf{x}_k)}(\mathbf{x}_k). \tag{1}$$

Then we have that

$$\begin{aligned} K &= \left(\bigcup_{k=1}^{j} B_{r(\mathbf{x}_k)}(\mathbf{x}_k)\right) \cap K && \text{by (1)} \\ &= \bigcup_{k=1}^{j} \left(B_{r(\mathbf{x}_k)}(\mathbf{x}_k) \cap K\right) \\ &= \bigcup_{k=1}^{j} \{\mathbf{x}_k\} && \text{by assumptions about } r(\mathbf{x}) \\ &= \{\mathbf{x}_1, \dots, \mathbf{x}_j\}. \end{aligned}$$

We conclude that K is a finite set. ☐

Proof using sequences. Assume to the contrary that K is infinite. Then we can construct a sequence $\{\mathbf{x}_k\}$ with $\mathbf{x}_k \in K$ for all $k \in \mathbb{N}$, and with $\mathbf{x}_k \neq \mathbf{x}_j$ for $k \neq j$.[1] Since K is compact, the sequence $\{\mathbf{x}_k\}$ has a convergent subsequence $\{\mathbf{x}_{j_k}\}$ with $\mathbf{x} := \lim_{k\to\infty} \mathbf{x}_{j_k} \in K$. Since elements of the original sequence are pairwise distinct, and j_k is strictly increasing, it follows that the elements of the subsequence are also pairwise distinct, i.e. $\mathbf{x}_{j_k} \neq \mathbf{x}_{j_\ell}$ if $k \neq \ell$.

Since $\mathbf{x} \in K$, we can by assumption find an $r(\mathbf{x}) > 0$ so that $B_{r(\mathbf{x})}(\mathbf{x}) \cap K = \{\mathbf{x}\}$. But, by the definition of $\lim_{k\to\infty} \mathbf{x}_{j_k}$, we can find an $N \in \mathbb{N}$ so that for $k \geq N$, $\mathbf{x}_{j_k} \in B_{r(\mathbf{x})}(\mathbf{x})$. Since $\mathbf{x}_{j_k} \in K$ for all $k \in \mathbb{N}$ this implies that

$$\mathbf{x}_{j_k} \in B_{r(\mathbf{x})}(\mathbf{x}) \cap K = \{\mathbf{x}\} \quad \text{for all } k \geq N.$$

We can conclude that $\mathbf{x}_{j_k} = \mathbf{x}$ for all $k \geq N$. This contradicts the fact that the $\mathbf{x}_{j_k}$ are pairwise distinct. Therefore K must be a finite set. ☐

[1] Such a sequence can be constructed as follows. Choose any $\mathbf{x}_1 \in K$, and inductively choose $\mathbf{x}_{k+1} \in K \setminus \{\mathbf{x}_1, \dots, \mathbf{x}_k\}$. If there exists a $k \in \mathbb{N}$ for which $K \setminus \{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is empty, then K would not be an infinite set.

(4) Let $K \subset \mathbb{R}^n$ be a compact set, let $F \subset \mathbb{R}^n$ be a closed set, and assume that $K \cap F = \emptyset$. Prove that there exists an open set O and a closed set C satisfying

$$K \subset O \subset C \subset F^c.$$

Proof. Let $\mathbf{x} \in K$. Since $K \cap F = \emptyset$, we can conclude that $\mathbf{x} \in F^c$. Since F is closed, F^c is open. Therefore, there exists an $\varepsilon(\mathbf{x}) > 0$ so that $B_{\varepsilon(\mathbf{x})}(\mathbf{x}) \subset F^c$. Choosing such an $\varepsilon(\mathbf{x}) > 0$ for each $\mathbf{x} \in K$, we have that $\left\{B_{\varepsilon(\mathbf{x})/2}(\mathbf{x})\right\}_{\mathbf{x} \in K}$ is an open cover for K since

$$K \subset \bigcup_{\mathbf{x} \in K} B_{\varepsilon(\mathbf{x})/2}(\mathbf{x}).$$

Since K is compact, we can find a finite subcover, i.e. there is a finite collection of points $\{\mathbf{x}_k\}_{k=1}^{j} \subset K$ so that

$$K \subset \bigcup_{k=1}^{j} B_{\varepsilon(\mathbf{x}_k)/2}(\mathbf{x}_k).$$

Since for any $\mathbf{x} \in K$ we have that

$$B_{\varepsilon(\mathbf{x})/2}(\mathbf{x}) \subset \bar{B}_{\varepsilon(\mathbf{x})/2}(\mathbf{x}) \subset B_{\varepsilon(\mathbf{x})}(\mathbf{x}) \subset F^c,$$

we can conclude that

$$K \subset \bigcup_{k=1}^{j} B_{\varepsilon(\mathbf{x}_k)/2}(\mathbf{x}_k) \subset \bigcup_{k=1}^{j} \bar{B}_{\varepsilon(\mathbf{x}_k)/2}(\mathbf{x}_k) \subset F^c.$$

Since any union of open sets is open (Theorem 8.24i), we can choose $O = \cup_{k=1}^{j} B_{\varepsilon(\mathbf{x}_k)/2}(\mathbf{x}_k)$, and since finite unions of closed sets are closed (Theorem 8.24iv), we can choose $C = \cup_{k=1}^{j} \bar{B}_{\varepsilon(\mathbf{x}_k)/2}(\mathbf{x}_k)$. $\square$