Math 421, Homework #7
Due: Wednesday, March 31

Homework assignments should be submitted during lecture. Please consult the homework rules and guidelines before completing the assignment. In particular:

- Write in compete sentences, organized into paragraphs.
- Leave plenty of room in between problems.
- Write only on the front side of each sheet of paper.
- Staple!
- Write on your assignment the names of any persons or sources consulted during its completion (other than the course text or instructor).

(1) Let $\{\mathbf{x}_k\}$ and $\{\mathbf{y}_k\}$ be convergent sequences in $\mathbb{R}^n$, and assume that $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{L}$ and that $\lim_{k\to\infty} \mathbf{y}_k = \mathbf{M}$. Prove directly from definition 9.1 (i.e. don't use Theorem 9.2) that:
  (a) $\lim_{k\to\infty} \mathbf{x}_k + \mathbf{y}_k = \mathbf{L} + \mathbf{M}$.
  (b) $\lim_{k\to\infty} \mathbf{x}_k \cdot \mathbf{y}_k = \mathbf{L} \cdot \mathbf{M}$.

(2) Prove directly from definition 9.10 (i.e. don't use the Heine-Borel Theorem or the Borel Covering Lemma) that if K_1 and K_2 are compact sets, then the union $K_1 \cup K_2$ is also compact.

(3) (9.2.4) Suppose that $K \subset \mathbb{R}^n$ is compact and that for every $\mathbf{x} \in K$ there is an $r(\mathbf{x}) > 0$ so that $B_{r(\mathbf{x})}(\mathbf{x}) \cap K = \{\mathbf{x}\}$. Prove that K is a finite set.

(4) Let $K \subset \mathbb{R}^n$ be a compact set, let $F \subset \mathbb{R}^n$ be a closed set, and assume that $K \cap F = \emptyset$. Prove that there exists an open set O and a closed set C satisfying

$$K \subset O \subset C \subset F^c.$$