Math 421, Homework #6 Solutions

(1) Let $E \subset \mathbb{R}^n$ Show that

$$\left(\bar{E}\right)^c = (E^c)^o\,,$$

i.e. the complement of the closure is the interior of the complement.[1]

Proof. Before giving the proof we recall characterizations of the interior and closure (proved in lecture) that will be useful: if S is any subset of $\mathbb{R}^n$ we have that

$$x \in S^o \iff \exists \varepsilon > 0, \text{ so that } B_\varepsilon(x) \subset S \tag{1}$$

and

$$x \in \bar{S} \iff \forall \varepsilon > 0, B_\varepsilon(x) \cap S \neq \emptyset. \tag{2}$$

Proceeding with the proof, we have that

$$\begin{aligned}
x \in \left(\bar{E}\right)^c &\iff x \notin \bar{E} && \text{by definition of complement}\\
&\iff \exists \varepsilon > 0 \text{ so that } B_\varepsilon(x) \cap E = \emptyset && \text{contrapositive of (2) with } S = E\\
&\iff \exists \varepsilon > 0 \text{ so that } B_\varepsilon(x) \subset E^c && \text{definitions of intersection and complement}\\
&\iff x \in (E^c)^o && \text{(1) applied with } S = E^c.
\end{aligned}$$

Therefore $\left(\bar{E}\right)^c = (E^c)^o$ as claimed. □

Alternate proof using the definitions. If $x \in (\bar{E})^c$, then by definition of closure, there exists a closed set $C \supset E$ so that $x \notin C$. But then $x \in C^c \subset E^c$, and since C is closed C^c is open. Therefore x is in an open subset of E^c and hence $x \in (E^c)^o$ by definition of interior.

Conversely, if $x \in (E^c)^o$, there exists an open set $V \subset E^c$ so that $x \in V$ by definition of interior. But then $x \notin V^c \supset E$. Since V is open, V^c is closed, so there exists a closed set V^c containing E with $x \notin V^c$. By definition of closure, this means that $x \notin \bar{E}$, and hence $x \in (\bar{E})^c$. □

Alternate proof using Theorem 8.32. Using Theorem 8.32i, $E \subset \bar{E}$. Using that complements reverse inclusions, this implies that $(\bar{E})^c \subset E^c$. Since $\bar{E}$ is closed, $(\bar{E})^c$ is an open set contained in E^c so, by Theorem 8.32ii applied to E^c, we can conclude that $(\bar{E})^c \subset (E^c)^o$.

Next, applying Theorem 8.32i to E^c we have that $(E^c)^o \subset E^c$, which implies that $E = (E^c)^c \subset ((E^c)^o)^c$. Since $(E^c)^o$ is open, we conclude that $((E^c)^o)^c$ is a closed set containing E so, by Theorem 8.32iii, we can conclude that $\bar{E} \subset ((E^c)^o)^c$. Taking complements in this last inclusion we get that $(E^c)^o = (((E^c)^o)^c)^c \subset (\bar{E})^c$. Combining this with the conclusion of the first paragraph, we conclude that $(E^c)^o = (\bar{E})^c$. □

[1] Note that each of the proofs given below can be adapted to show that $\overline{E^c} = (E^o)^c$, i.e. the closure of the complement is the complement of the interior.

(2) Let $E \subset \mathbb{R}^n$.

(a) Show that if E is connected, then the closure $\bar{E}$ is also connected.

Proof. Assume that $\overline{E}$ is not connected. Then there exist subsets U, V of $\overline{E}$ so that U and V are disjoint ($U \cap V = \emptyset$), nonempty, relatively open in $\overline{E}$, and so that $\overline{E} = U \cup V$.

Define $U' = E \cap U$, and $V' = E \cap V$. We claim that U' and V' are nonempty, relatively open in E, and satisfy $E = U' \cup V'$, and $U' \cap V' = \emptyset$, and thus E is not connected if $\overline{E}$ is not connected. Indeed, using that $E \subset \overline{E}$, we find

$$U' \cup V' = (E \cap U) \cup (E \cap V) = E \cap (U \cup V) = E \cap \overline{E} = E$$

and further

$$U' \cap V' = (E \cap U) \cap (E \cap V) = E \cap (U \cap V) = E \cap \emptyset = \emptyset,$$

so $U' \cup V' = E$ and $U' \cap V' = \emptyset$, as claimed.

To see that U' is relatively open in E, we note that since U is relatively open in $\overline{E}$, there exists an open set $A \subset \mathbb{R}^n$ so that $U = \overline{E} \cap A$. Then,

$$U' = E \cap U = E \cap (\overline{E} \cap A) = (E \cap \overline{E}) \cap A = E \cap A$$

so U' is relatively open in E since A is open. An identical argument shows that V' is relatively open in E.

Finally we claim that U' is nonempty. We saw above that there is an open set A so that $U' = E \cap A$ and $U = \overline{E} \cap A \neq \emptyset$. Suppose that $U' = E \cap A$ is empty. Then $E \subset A^c$, and since A is open A^c is closed. But since A^c is a closed set containing E, Theorem 8.32 (iii) tells us that $\overline{E} \subset A^c$. This in turn implies that $U = \overline{E} \cap A = \emptyset$, in contradiction to the fact that $U \neq \emptyset$. Therefore U' is nonempty. An identical arguement shows that $V' = E \cap V$ is nonempty since V is nonempty and relatively open in $\overline{E}$.

In conclusion, we have shown that if $\overline{E}$ is not connected, then E is not connected. Therefore, if E is connected, $\overline{E}$ must be connected as well. □

(b) Is the converse true, i.e. if $\bar{E}$ is connected must it be the case that E is also connected? Prove or find a counterexample.

No. A counterexample is given by $E = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$. The set is not connected since it is separated by the sets $U = (-\infty, 0)$ and $V = (0, +\infty)$. The closure of E is $\bar{E} = \mathbb{R}$ since $\mathbb{R}$ is the only closed set containing E. Theorem 8.30 shows that $\mathbb{R}$ is connected, so we have found an example of a set E which is not connected, but has connected closure. □

(3) (8.4.9) Find examples of:

(a) sets A, B in $\mathbb{R}$ such that $(A \cup B)^o \neq A^o \cup B^o$.

Example. Let $A = [-1, 0]$ and $B = [0, 1]$. Then

$$(A \cup B)^o = ([-1, 0] \cup [0, 1])^o = [-1, 1]^o = (-1, 1)$$

while

$$A^o \cup B^o = [-1, 0]^o \cup [0, 1]^o = (-1, 0) \cup (0, 1).$$

□

(b) sets A, B in $\mathbb{R}$ such that $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

Example. Let $A = (-1, 0)$ and $B = (0, 1)$. Then

$$\overline{A \cap B} = \overline{(-1, 0) \cap (0, 1)} = \overline{\emptyset} = \emptyset$$

while

$$\bar{A} \cap \bar{B} = \overline{(-1, 0)} \cap \overline{(0, 1)} = [-1, 0] \cap [0, 1] = \{0\}.$$

□

(c) sets A, B in $\mathbb{R}$ such that $\partial(A \cup B) \neq \partial A \cup \partial B$ and $\partial(A \cap B) \neq \partial A \cup \partial B$.

Example. Let $A = [-1, 0]$ and $B = [0, 1]$. Then

$$\partial A \cup \partial B = \{-1, 0\} \cup \{0, 1\} = \{-1, 0, 1\}$$

while

$$\partial(A \cup B) = \partial([-1, 1]) = \{-1, 1\}$$

and

$$\partial(A \cap B) = \partial(\{0\}) = \{0\}.$$

□

(4) (9.1.8)

(a) Let E be a subset of $\mathbb{R}^n$. A point $\mathbf{a} \in \mathbb{R}^n$ is called a *cluster point* of E if $E \cap B_r(\mathbf{a})$ contains infinitely many points for every $r > 0$. Prove that $\mathbf{a}$ is a cluster point of E if and only if for each $r > 0$, $E \cap B_r(\mathbf{a}) \setminus \{\mathbf{a}\}$ is nonempty.

Proof. First assume that $\mathbf{a}$ is a cluster point of E, i.e. that for every $r > 0$ the set $E \cap B_r(\mathbf{a})$ contains infinitely many points. We aim to show that for all $r > 0$ that $E \cap B_r(\mathbf{a}) \setminus \{\mathbf{a}\}$ is nonempty. Assuming to the contrary that there exists an $r' > 0$ so that

$$E \cap B_{r'}(\mathbf{a}) \setminus \{\mathbf{a}\} = \emptyset$$

lets us conclude that

$$E \cap B_{r'}(\mathbf{a}) \subset \{\mathbf{a}\}.$$

Thus $E \cap B_{r'}(\mathbf{a})$ is a finite set in contradiction to the assumption that $E \cap B_r(\mathbf{a})$ contains infinitely many points for all $r > 0$. We conclude that $E \cap B_r(\mathbf{a}) \setminus \{\mathbf{a}\}$ is nonempty for all $r > 0$.

We next assume that $E \cap B_r(\mathbf{a}) \setminus \{\mathbf{a}\}$ is nonempty for all $r > 0$. Define $r_1 = 1$ and choose a point $\mathbf{x}_1 \in E \cap B_{r_1}(\mathbf{a}) \setminus \{\mathbf{a}\}$. Then construct sequences $\{r_k\}$ and $\{\mathbf{x}_k\}$ inductively by

$$r_k = \min\left\{\tfrac{1}{2^{k-1}}, \|\mathbf{x}_{k-1} - \mathbf{a}\|\right\}$$

and choose an $\mathbf{x}_k \in E \cap B_{r_k}(\mathbf{a}) \setminus \{\mathbf{a}\}$. Note that each r_k is positive since $\mathbf{x}_k \neq \mathbf{a}$ by assumption. Further, note that $\mathbf{x}_k \neq \mathbf{x}_{k-1}$ and $r_k < r_{k-1}$ for all $k > 1$ since

$$\|\mathbf{x}_k - \mathbf{a}\| < r_k \leq \|\mathbf{x}_{k-1} - \mathbf{a}\| < r_{k-1} \tag{3}$$

by construction. Finally, note that the squeeze theorem tells us that $\lim_{k\to\infty} r_k = 0$ since

$$0 < r_k \leq \tfrac{1}{2^{k-1}} \text{ for all } k \in \mathbb{N} \setminus \{1\}.$$

We claim that for each $r > 0$ that $E \cap B_r(\mathbf{a})$ contains infinitely many points. Indeed, given $r > 0$, the fact that $\lim_{k\to\infty} r_k = 0$ allows us to find an $N \in \mathbb{N}$ so that $r_k < r$ for $k \geq N$. Then for $k \geq N$ we can use (3) to conclude that

$$\|\mathbf{x}_k - \mathbf{a}\| < r_k \leq r_N < r$$

so $\mathbf{x}_k \in B_r(\mathbf{a})$ for all $k \geq N$. Since $\mathbf{x}_k \in E$ by construction and each of the $\mathbf{x}_k$ are distinct by (3), we can conclude that $B_r(\mathbf{a}) \cap E$ contains the infinite set $\{\mathbf{x}_k\}_{k \geq N}$. □

Alternate proof. We argue exactly as above to show that if $\mathbf{a}$ is a cluster point of E, then $E \cap B_r(\mathbf{a}) \setminus \{\mathbf{a}\}$ is nonempty for all $r > 0$.

We next assume that $E \cap B_r(\mathbf{a}) \setminus \{\mathbf{a}\}$ is nonempty for all $r > 0$. We will argue by contradiction to show that $\mathbf{a}$ is a cluster point of E. If $\mathbf{a}$ is not cluster point of E, then there exists an $r' > 0$ for which $E \cap B_{r'}(\mathbf{a})$ is a finite set. We can conclude that $E \cap B_{r'}(\mathbf{a}) \setminus \{\mathbf{a}\}$ is also a finite set, and we write $E \cap B_{r'}(\mathbf{a}) \setminus \{\mathbf{a}\} = \{\mathbf{x}_1, \ldots, \mathbf{x}_j\}$. Define

$$r'' = \min_{i \in \{1,\ldots,j\}} \{\|\mathbf{x}_i - \mathbf{a}\|\} > 0.$$

Then, since $\|\mathbf{x}_i - \mathbf{a}\| \geq r''$ for all $i \in \{1, \ldots, j\}$, it follows that $E \cap B_{r''}(\mathbf{a}) \setminus \{\mathbf{a}\}$ is empty, in contradiction to the assumption that $E \cap B_r(\mathbf{a}) \setminus \{\mathbf{a}\}$ is nonempty for all $r > 0$. This contradiction shows that $\mathbf{a}$ must be a cluster point of E. □

(b) Prove that every bounded infinite subset of $\mathbb{R}^n$ has at least one cluster point.

Proof. Denoting the set by E, the fact that E is infinite allows us to choose a sequence $\{\mathbf{x}_k\}_{k\in\mathbb{N}}$ with $\mathbf{x}_k \in E$ for all $k \in \mathbb{N}$ and $\mathbf{x}_k \neq \mathbf{x}_j$ if $k \neq j$.[2] By the Bolzano-Weierstrass theorem $\{\mathbf{x}_k\}$ has a convergent subsequence which we will denote by $\mathbf{y}_k = \mathbf{x}_{j_k}$, and we note that since the $\mathbf{x}_k$ are distinct, so are the $\mathbf{y}_k$, i.e. $\mathbf{y}_k \neq \mathbf{y}_j$ if $k \neq j$. Let $\mathbf{L} = \lim_{k\to\infty} \mathbf{y}_k$. We claim that $\mathbf{L}$ is a cluster point of E. Indeed, let $r > 0$. Since $\mathbf{y}_k$ converges to $\mathbf{L}$, there is an $N \in \mathbb{N}$ so that $\mathbf{y}_k \in B_r(\mathbf{L})$ if $k \geq N$. Since the $\mathbf{y}_k$ are distinct, and $\mathbf{y}_k \in E$ for all $k \in \mathbb{N}$ by assumption, we conclude that $B_r(\mathbf{L}) \cap E$ contains the infinite set $\{\mathbf{y}_k\}_{k\geq N}$. Therefore $\mathbf{L}$ is a cluster point of E as claimed. □

[2] Choose any $\mathbf{x}_1 \in E$, and then choose

$$\mathbf{x}_{n+1} \in E \setminus \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}.$$

Since E is assumed to infinite, none of the sets $E \setminus \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ can be empty.