Math 421, Homework #6
Due: Wednesday, March 17

Homework assignments should be submitted during lecture. Please consult the homework rules and guidelines before completing the assignment. In particular:

- Write in compete sentences, organized into paragraphs.
- Leave plenty of room in between problems.
- Write only on the front side of each sheet of paper.
- Staple!
- Write on your assignment the names of any persons or sources consulted during its completion (other than the course text or instructor).

(1) Let $E \subset \mathbb{R}^n$ Show that
$$\left(\bar{E}\right)^c = (E^c)^o \,,$$
i.e. the complement of the closure is the interior of the complement.

(2) Let $E \subset \mathbb{R}^n$.
  (a) Show that if E is connected, then the closure $\bar{E}$ is also connected.

  (b) Is the converse true, i.e. if $\bar{E}$ is connected must it be the case that E is also connected? Prove or find a counterexample.

(3) (8.4.9) Find examples of:
  (a) sets A, B in $\mathbb{R}$ such that $(A \cup B)^o \neq A^o \cup B^o$.

  (b) sets A, B in $\mathbb{R}$ such that $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

  (c) sets A, B in $\mathbb{R}$ such that $\partial(A \cup B) \neq \partial A \cup \partial B$ and $\partial(A \cap B) \neq \partial A \cup \partial B$.

(4) (9.1.8)
  (a) Let E be a subset of $\mathbb{R}^n$. A point $\mathbf{a} \in \mathbb{R}^n$ is called a *cluster point* of E if $E \cap B_r(\mathbf{a})$ contains infinitely many points for every $r > 0$. Prove that $\mathbf{a}$ is a cluster point of E if and only if for each $r > 0$, $E \cap B_r(\mathbf{a}) \setminus \{\mathbf{a}\}$ is nonempty.

  (b) Prove that every bounded[1] infinite subset of $\mathbb{R}^n$ has at least one cluster point.

[1] A subset A of $\mathbb{R}^n$ is said to be bounded if there is an $R > 0$ so that $A \subset B_R(\mathbf{0})$.