Math 421, Homework #5 Solutions

(1) (8.3.6) Suppose that $E \subset \mathbb{R}^n$ and C is a subset of E.

(a) Prove that if E is closed, then C is relatively closed in E if and only if C is a closed set (as defined in Definition 8.20(ii)).

Proof. First assume that C is a closed set. Then since $C \subset E$ we have that

$$C = E \cap C$$

so C can be written as the intersection of a closed set with E. Therefore C is relatively closed in E.

Next assume that C is relatively closed in E. Then, there exists a closed set B so that

$$C = E \cap B.$$

Since E is assumed to be closed, this means that C is the intersection of two closed sets, so C is a closed set since intersections of closed sets are closed. □

(b) Prove that C is relatively closed in E if and only if $E \setminus C$ is relatively open in E.

Proof. Assume that C is relatively closed in E. Then, there exists a closed set B so that

$$C = E \cap B.$$

We then find that

$$\begin{aligned} E \setminus C &= E \cap C^c \\ &= E \cap (E \cap B)^c \\ &= E \cap (E^c \cup B^c) \\ &= (E \cap E^c) \cup (E \cap B^c) \\ &= \emptyset \cup (E \cap B^c) = E \cap B^c. \end{aligned}$$

Since B is closed, B^c is open. Therefore $E \setminus C$ is relatively open in E since it can be written as E intersected with an open set.

Next assume that $E \setminus C$ is relatively open in E. Then there exists an open set A so that

$$E \setminus C = E \cap A. \tag{1}$$

We claim that $C = E \cap A^c$. Indeed since $C \subset E$ we have that $E \cap C = C$ and thus

$$\begin{aligned} E \setminus (E \setminus C) &= E \setminus (E \cap C^c) \\ &= E \cap (E \cap C^c)^c \\ &= E \cap (E^c \cup C) \\ &= (E \cap E^c) \cup (E \cap C) \\ &= \emptyset \cup C = C. \end{aligned}$$

Meanwhile, (1) gives us

$$\begin{aligned} E \setminus (E \setminus C) &= E \setminus (E \cap A) \\ &= E \cap (E \cap A)^c \\ &= E \cap (E^c \cup A^c) \\ &= (E \cap E^c) \cup (E \cap A^c) \\ &= \emptyset \cup (E \cap A^c) \\ &= E \cap A^c. \end{aligned}$$

Combining the above gives $C = E \cap A^c$ as claimed. Since A is open, A^c is closed, so $C = E \cap A^c$ implies that C is relatively closed in E.

□

The following lemma will be useful in parts (a) and (b) of the next problem.

Lemma. *Let $E \subset \mathbb{R}^n$, and assume that $U \subset E$ is relatively open in E. Then for any subset E' of E, $U \cap E'$ is relatively open in E'.*

Proof. By the assumption that $U \subset E$ is relatively open in E, there is an open set A so that

$$U = E \cap A.$$

Since $E' \subset E$, $E' \cap E = E'$ so

$$U \cap E' = (E \cap A) \cap E' = (E \cap E') \cap A = E' \cap A.$$

Therefore $U \cap E'$ can be written as E' intersected with an open set, so $U \cap E'$ is relatively open in E'. □

(2) (8.3.7)

(a) If A and B are connected subsets of $\mathbb{R}^n$ and $A \cap B \neq \emptyset$, prove that $A \cup B$ is connected.

Proof. Assume that $A \cup B$ is not connected. Then there exist nonempty sets $U \subset A \cup B$ and $V \subset A \cup B$ satisfying

- U and V are relatively open in $A \cup B$,
- $U \cap V = \emptyset$, and
- $A \cup B = U \cup V$.

Let $x \in A \cap B$ (which we are assuming is nonempty). Then either $x \in U$ or $x \in V$. Without loss of generality assume that $x \in U$. Then both $U \cap A$ and $U \cap B$ are nonempty since both contain x. Since $V \subset A \cup B$ is assumed to be nonempty, we know that V intersects at least one of A or B, and without loss of generality assume that $V \cap A$ is nonempty.

Define sets $U' = U \cap A$ and $V' = V \cap A$, and note that both U' and V' are nonempty. Since $A \subset A \cup B$ the lemma above implies that U' and V' are relatively open in A. Moreover,

$$U' \cap V' = (U \cap A) \cap (V \cap A) = (U \cap V) \cap A = \emptyset \cap A = \emptyset,$$

and

$$U' \cup V' = (U \cap A) \cup (V \cap A) = (U \cup V) \cap A = (A \cup B) \cap A = A.$$

Thus U' and V' separate A, which contradicts that assumption that A is a connected set. We can thus conclude that $A \cup B$ is connected. □

(b) If $\{E_\alpha\}_{\alpha\in A}$ is a collection of connected sets in $\mathbb{R}^n$ and $\cap_{\alpha\in A}E_\alpha \neq \emptyset$, prove that

$$E = \cup_{\alpha\in A}E_\alpha$$

is connected.

Proof. Assume that E is not connected. Then there exist sets $U \subset E$ and $V \subset E$, which are nonempty, disjoint ($U \cap V = \emptyset$), relatively open in E, with $E = U \cup V$. Let $x \in \cap_{\alpha\in A}E_\alpha$ (which we are assuming is nonempty). Then $x \in E = U \cup V$ so $x \in U$ or $x \in V$, and without loss of generality, we can assume that $x \in U$. Therefore

$$U \cap E_\alpha \neq \emptyset \qquad \text{for all } \alpha \in A \tag{2}$$

since each of these sets contains x. Since $V \subset E$ we have that

$$V = V \cap E = V \cap (\cup_{\alpha\in A}E_\alpha) = \cup_{\alpha\in A}(V \cap E_\alpha).$$

Since V is nonempty, this implies that there exist an $\alpha' \in A$ so that

$$V \cap E_{\alpha'} \neq \emptyset. \tag{3}$$

We claim that $E_{\alpha'}$ is not connected, which would be a contradiction. Define $U' = U \cap E_{\alpha'}$ and $V' = V \cap E_{\alpha'}$. Both U' and V' are nonempty by (2) and (3). Moreover, the lemma above implies that U' and V' are relatively open in $E_{\alpha'}$ since $E_{\alpha'} \subset E$ by definition of E. Moreover

$$U' \cap V' = (U \cap E_{\alpha'}) \cap (V \cap E_{\alpha'}) = (U \cap V) \cap E_{\alpha'} = \emptyset \cap E_{\alpha'} = \emptyset$$

so $U' \cap V' = \emptyset$. Furthermore, since $E_{\alpha'} \subset E$, $E_{\alpha'} \cap E = E_{\alpha'}$ so

$$U' \cup V' = (U \cap E_{\alpha'}) \cup (V \cap E_{\alpha'}) = (U \cup V) \cap E_{\alpha'} = E \cap E_{\alpha'} = E_{\alpha'},$$

so $U' \cup V' = E_{\alpha'}$. Therefore $E_{\alpha'}$ is not connected, which contradicts our assumptions. This contradiction let's us conclude that E is connected. □

(c) If A and B are connected subset of $\mathbb{R}$ and $A \cap B \neq \emptyset$, prove that $A \cap B$ is connected.

Proof sketch 1. Theorem 8.30 tells us that $A \cap B$ are intervals, i.e. sets of one of the following forms:

$$\begin{aligned} &\{x \in \mathbb{R} \mid a < x < b\} && \text{with } -\infty \leq a \text{ and } b \leq +\infty \\ &\{x \in \mathbb{R} \mid a \leq x < b\} && \text{with } -\infty < a < b \leq +\infty \\ &\{x \in \mathbb{R} \mid a < x \leq b\} && \text{with } -\infty \leq a < b < +\infty \\ &\{x \in \mathbb{R} \mid a \leq x \leq b\} && \text{with } -\infty < a \leq b < +\infty. \end{aligned}$$

A straightforward, but arduous argument by cases shows that any two sets of one of the above form have intersection with one of the above forms. Therefore $A \cap B$ is an interval and hence connected by Theorem 8.30. □

Proof 2. Let $x_0 \in A \cap B$ (which is assumed to be nonempty). Define

$$I = \{[a, b] \subset \mathbb{R} \mid x_0 \in [a, b] \subset A \cap B\}$$

i.e. an element of I is a (possibly degenerate) closed interval containing x_0 and contained in $A \cap B$. Note that by definition

$$x_0 \in \bigcap_{[a,b] \in I} [a, b] \neq \emptyset.$$

Since intervals are connected by Theorem 8.30, part (b) let's us conclude that

$$E := \bigcup_{[a,b] \in I} [a, b] \tag{4}$$

is connected.

We claim that $E = A \cap B$, which will finish the proof. Indeed, from the definition of E we have that $E \subset A \cap B$ since each interval on the right hand side of (4) is assumed to be a subset $A \cap B$. Let $c \in A \cap B$. If $c > x_0$ then we claim that $[x_0, c] \subset A \cap B$. If not, then there would be a $d \in (x_0, c)$ with either $d \notin A$ or $d \notin B$. Without loss of generality, assume that $d \notin A$. Then $A \cap (-\infty, d)$ and $A \cap (d, +\infty)$ are nonempty (since the first contains x_0 and the second contains c) relatively open sets which separate A in contradiction to the assumption that A is connected. We conclude that $[x_0, c] \subset A \cap B$ which implies that $[x_0, c] \in I$ and hence that $c \in E$. Similarly, we can argue that if $c \leq x_0$, then $[c, x_0] \subset A \cap B$ (or else either A or B wouldn't be connected) so $[c, x_0] \in I$ and hence $c \in E$. Hence $A \cap B \subset E$. Thus $A \cap B = E$ as claimed and therefore $A \cap B$ is connected. □

(d) Show that part (c) is no longer true if $\mathbb{R}^2$ replaces $\mathbb{R}$, i.e. provide an example of a pair of connected sets in $\mathbb{R}^2$ whose intersection is not connected. (A clearly drawn picture and explanation of your picture would be a sufficient answer here.)

Example. Let

$$A = \{(\cos t, \sin t)] \mid t \in [0, \pi]\}$$

and

$$B = \{(\cos t, \sin t) \mid t \in [\pi, 2\pi]\}\,.$$

Then A and B are connected because they are the continuous image of intervals (we will prove this in class eventually), but

$$A \cap B = \{(-1, 0), (1, 0)\}$$

which is not a connected set since $U = \{(-1, 0)\}$ and $V = \{(1, 0)\}$ are nonempty, disjoint, relatively open in $A \cap B$ and $A \cap B = U \cup V$. □

(3) (8.3.8) Let V be a subset of $\mathbb{R}^n$.

(a) Prove that V is open if and only if there is a collection of open balls $\{B_\alpha\}_{\alpha\in A}$ such that

$$V = \cup_{\alpha\in A} B_\alpha.$$

Proof. Assume that V is open. Then for any $x \in V$ there is an $\varepsilon_x > 0$ so that $B_{\varepsilon_x}(x) \subset V$. We claim that

$$V = \cup_{x\in V} B_{\varepsilon_x}(x)$$

Indeed if $y \in V$ then $y \in B_{\varepsilon_y}(y)$ so $y \in \cup_{x\in V} B_{\varepsilon_x}(x)$ and hence $V \subset \cup_{x\in V} B_{\varepsilon_x}(x)$. Conversely, since $B_{\varepsilon_x}(x) \subset V$ for all $x \in V$ it follows that $\cup_{x\in V} B_{\varepsilon_x}(x) \subset V$. Therefore $V = \cup_{x\in V} B_{\varepsilon_x}(x)$, so V can be written as a union of open balls.

Now assume that there is a collection of open balls $\{B_\alpha\}_{\alpha\in A}$ so that

$$V = \cup_{\alpha\in A} B_\alpha.$$

Then since any union of open sets is open, we can conclude that V is open. □

(b) What happens to this result if *open* is replaced by *closed*, i.e. is it true that a set is closed if and only if it can be written as a union of closed balls? Prove or provide a counterexample.

Answer. Since a set with a single point is a closed ball of radius zero we can write any closed set C as

$$C = \bigcup_{x\in C} \overline{B}_0(x),$$

and hence every closed set is a union of closed balls.

However, since

$$(-1,1) = \bigcup_{n\in\mathbb{N}} [-1+\tfrac{1}{n}, 1-\tfrac{1}{n}] = \bigcup_{n\in\mathbb{N}} \overline{B}_{1-1/n}(0),$$

and $(-1,1)$ is not closed, not every union of closed balls is a closed set. □