Math 421, Homework #5
Due: Wednesday, March 3

Homework assignments should be submitted during lecture. Please consult the homework rules and guidelines before completing the assignment. In particular:

- Write in compete sentences, organized into paragraphs.
- Leave plenty of room in between problems.
- Write only on the front side of each sheet of paper.
- Staple!
- Write on your assignment the names of any persons or sources consulted during its completion (other than the course text or instructor).

(1) (8.3.6) Suppose that $E \subset \mathbb{R}^n$ and C is a subset of E.
  (a) Prove that if E is closed, then C is relatively closed in E if and only if C is a closed set (as defined in Definition 8.20(ii)).
  (b) Prove that C is relatively closed in E if and only if $E \setminus C$ is relatively open in E.

(2) (8.3.7)
  (a) If A and B are connected subsets of $\mathbb{R}^n$ and $A \cap B \neq \emptyset$, prove that $A \cup B$ is connected.
  (b) If $\{E_\alpha\}_{\alpha \in A}$ is a collection of connected sets in $\mathbb{R}^n$ and $\cap_{\alpha \in A} E_\alpha \neq \emptyset$, prove that
$$E = \cup_{\alpha \in A} E_\alpha$$
  is connected.
  (c) If A and B are connected subset of $\mathbb{R}$ and $A \cap B \neq \emptyset$, prove that $A \cap B$ is connected.
  (d) Show that part (c) is no longer true if $\mathbb{R}^2$ replaces $\mathbb{R}$, i.e. provide an example of a pair of connected sets in $\mathbb{R}^2$ whose intersection is not connected. (A clearly drawn picture and explanation of your picture would be a sufficient answer here.)

(3) (8.3.8) Let V be a subset of $\mathbb{R}^n$.
  (a) Prove that V is open if and only if there is a collection of open balls $\{B_\alpha\}_{\alpha \in A}$ such that
$$V = \cup_{\alpha \in A} B_\alpha.$$
  (b) What happens to this result if *open* is replaced by *closed*, i.e. is it true that a set is closed if and only if it can be written as a union of closed balls? Prove or provide a counterexample.