Math 421, Homework #4 Solutions

Note: In problems (1) and (2) we will denote the operator norm by $\|\cdot\|_{\mathcal{L}}$ to distinguish it from the usual norm $\|\cdot\|$ on $\mathbb{R}^n$.

(1) (8.2.11) Let $\mathbf{T} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, and define

$$M_1 := \sup_{\|\mathbf{x}\|=1} \|\mathbf{T}(\mathbf{x})\| = \sup\left\{\|\mathbf{T}(\mathbf{x})\| \,\middle|\, \mathbf{x} \in \mathbb{R}^n; \|\mathbf{x}\| = 1\right\}$$

$$M_2 := \inf\left\{C > 0 \,\middle|\, \|\mathbf{T}(\mathbf{x})\| \le C\|\mathbf{x}\| \text{ for all } \mathbf{x} \in \mathbb{R}^n\right\}.$$

(a) Prove that $M_1 \le \|\mathbf{T}\|_{\mathcal{L}}$.

Proof. We claim that

$$\left\{\|\mathbf{T}(\mathbf{x})\| \,\middle|\, \mathbf{x} \in \mathbb{R}^n; \|\mathbf{x}\| = 1\right\} \subset \left\{\tfrac{\|\mathbf{T}(\mathbf{x})\|}{\|\mathbf{x}\|} \,\middle|\, \mathbf{x} \neq \mathbf{0}\right\}$$ [1]

Indeed if $\|\mathbf{x}\| = 1$ then $\mathbf{x} \neq \mathbf{0}$ so

$$\|\mathbf{T}(\mathbf{x})\| = \tfrac{\|\mathbf{T}(\mathbf{x})\|}{1} = \tfrac{\|\mathbf{T}(\mathbf{x})\|}{\|\mathbf{x}\|} \in \left\{\tfrac{\|\mathbf{T}(\mathbf{x})\|}{\|\mathbf{x}\|} \,\middle|\, \mathbf{x} \neq \mathbf{0}\right\}.$$

It then follows from properties of suprema that

$$M_1 := \sup\left\{\|\mathbf{T}(\mathbf{x})\| \,\middle|\, \mathbf{x} \in \mathbb{R}^n; \|\mathbf{x}\| = 1\right\} \le \sup\left\{\tfrac{\|\mathbf{T}(\mathbf{x})\|}{\|\mathbf{x}\|} \,\middle|\, \mathbf{x} \neq \mathbf{0}\right\} =: \|\mathbf{T}\|_{\mathcal{L}}.$$

□

(b) Using the linear property of $\mathbf{T}$, prove that if $\mathbf{x} \neq \mathbf{0}$, then

$$\frac{\|\mathbf{T}(\mathbf{x})\|}{\|\mathbf{x}\|} \le M_1.$$

Proof. Let $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, and define $\mathbf{y} = \frac{1}{\|\mathbf{x}\|}\mathbf{x}$. Using the properties of the norm, we then have that

$$\|\mathbf{y}\| = \left\|\tfrac{1}{\|\mathbf{x}\|}\mathbf{x}\right\| = \left|\tfrac{1}{\|\mathbf{x}\|}\right| \|\mathbf{x}\| = \tfrac{1}{\|\mathbf{x}\|}\|\mathbf{x}\| = 1.$$

We can then conclude that

$$\|\mathbf{T}(\mathbf{y})\| \le \sup\left\{\|\mathbf{T}(\mathbf{x})\| \,\middle|\, \mathbf{x} \in \mathbb{R}^n; \|\mathbf{x}\| = 1\right\} =: M_1.$$

Using the linearity of $\mathbf{T}$ and the properties of norms, we also have that

$$\|\mathbf{T}(\mathbf{y})\| = \left\|\mathbf{T}(\tfrac{1}{\|\mathbf{x}\|}\mathbf{x})\right\| = \left\|\tfrac{1}{\|\mathbf{x}\|}\mathbf{T}(\mathbf{x})\right\| = \left|\tfrac{1}{\|\mathbf{x}\|}\right| \|\mathbf{T}(\mathbf{x})\| = \tfrac{1}{\|\mathbf{x}\|}\|\mathbf{T}(\mathbf{x})\|.$$

Combining the previous two lines then gives us $\frac{\|\mathbf{T}(\mathbf{x})\|}{\|\mathbf{x}\|} \le M_1$. □

[1] Using the linearity of $\mathbf{T}$ one can actually show that these sets are equal, which gives a slightly different proof that $\|\mathbf{T}\|_{\mathcal{L}} = M_1$.

(c) Prove that $M_1 = M_2 = \|\mathbf{T}\|_{\mathcal{L}}$.

Proof. We first show that $M_1 = \|\mathbf{T}\|_{\mathcal{L}}$. In part (b) we showed if $\mathbf{x} \neq \mathbf{0}$ then $M_1 \geq \frac{\|\mathbf{T}(\mathbf{x})\|}{\|\mathbf{x}\|}$. This shows that M_1 is an upper bound for the set

$$\left\{ \tfrac{\|\mathbf{T}(\mathbf{x})\|}{\|\mathbf{x}\|} \,\middle|\, \mathbf{x} \neq \mathbf{0} \right\}$$

and consequently that

$$M_1 \geq \sup \left\{ \tfrac{\|\mathbf{T}(\mathbf{x})\|}{\|\mathbf{x}\|} \,\middle|\, \mathbf{x} \neq \mathbf{0} \right\} =: \|\mathbf{T}\|_{\mathcal{L}}\,.$$

Since we showed in part (a) that $M_1 \leq \|\mathbf{T}\|_{\mathcal{L}}$ we can conclude that $\|\mathbf{T}\|_{\mathcal{L}} = M_1$.

We next show that $M_2 = \|\mathbf{T}\|_{\mathcal{L}}$. We start by showing that $\|\mathbf{T}\|_{\mathcal{L}} \leq M_2$. Our strategy will be to show that $\|\mathbf{T}\|_{\mathcal{L}}$ is a lower bound for the set

$$S := \left\{ C > 0 \,\middle|\, \|\mathbf{T}(\mathbf{x})\| \leq C\|\mathbf{x}\| \text{ for all } \mathbf{x} \in \mathbb{R}^n \right\}$$

which would then let us conclude that $\|\mathbf{T}\|_{\mathcal{L}} \leq \inf S =: M_2$. Let $c < \|\mathbf{T}\|_{\mathcal{L}}$. Then, defining $\varepsilon = \|\mathbf{T}\|_{\mathcal{L}} - c > 0$, we can use the definition of $\|\mathbf{T}\|_{\mathcal{L}}$ and the approximation property of suprema to find an $\mathbf{x}_c \in \mathbb{R}^n \setminus \{0\}$ satisfying

$$\frac{\|\mathbf{T}(\mathbf{x}_c)\|}{\|\mathbf{x}_c\|} > \|\mathbf{T}\|_{\mathcal{L}} - \varepsilon = c$$

or equivalently $\|\mathbf{T}(\mathbf{x}_c)\| > c\,\|\mathbf{x}_c\|$. We conclude that if $c < \|\mathbf{T}\|_{\mathcal{L}}$ then $c \notin S$. Thus, if $c \in S$ it must be the case that $c \geq \|\mathbf{T}\|_{\mathcal{L}}$, which means that $\|\mathbf{T}\|_{\mathcal{L}}$ is a lower bound for S as claimed. As observed above, this lets us conclude that

$$\|\mathbf{T}\|_{\mathcal{L}} \leq \inf S = \inf \left\{ C > 0 \,\middle|\, \|\mathbf{T}(\mathbf{x})\| \leq C\|\mathbf{x}\| \text{ for all } \mathbf{x} \in \mathbb{R}^n \right\} =: M_2.$$

To finish the proof, we next show that $\|\mathbf{T}\|_{\mathcal{L}} \geq M_2$. To do this we observe that since Theorem 8.16 tells us that

$$\|\mathbf{T}(\mathbf{x})\| \leq \|\mathbf{T}\|_{\mathcal{L}}\,\|\mathbf{x}\| \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

it follows that

$$\|\mathbf{T}\|_{\mathcal{L}} \in \left\{ C > 0 \,\middle|\, \|\mathbf{T}(\mathbf{x})\| \leq C\|\mathbf{x}\| \text{ for all } \mathbf{x} \in \mathbb{R}^n \right\}$$

and consequently that

$$\|\mathbf{T}\|_{\mathcal{L}} \geq \inf \left\{ C > 0 \,\middle|\, \|\mathbf{T}(\mathbf{x})\| \leq C\|\mathbf{x}\| \text{ for all } \mathbf{x} \in \mathbb{R}^n \right\} =: M_2.$$

We conclude that $\|\mathbf{T}\|_{\mathcal{L}} = M_2$ as claimed. $\square$

(2) Let $\mathbf{T}$, $\mathbf{U} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.[2]

(a) Prove that
$$\|\mathbf{T} + \mathbf{U}\|_{\mathcal{L}} \leq \|\mathbf{T}\|_{\mathcal{L}} + \|\mathbf{U}\|_{\mathcal{L}}$$
where $\mathbf{T} + \mathbf{U}$ is the linear transformation defined by $(\mathbf{T} + \mathbf{U})(\mathbf{x}) = \mathbf{T}(\mathbf{x}) + \mathbf{U}(\mathbf{x})$.

Proof. Given $\mathbf{x} \in \mathbb{R}^n$ we have that
$$\begin{aligned}
\|(\mathbf{T} + \mathbf{U})(\mathbf{x})\| &= \|\mathbf{T}(\mathbf{x}) + \mathbf{U}(\mathbf{x})\| && \text{by definition of } \mathbf{T} + \mathbf{U} \\
&\leq \|\mathbf{T}(\mathbf{x})\| + \|\mathbf{U}(\mathbf{x})\| && \text{by the triangle inequality} \\
&\leq \|\mathbf{T}\|_{\mathcal{L}} \|\mathbf{x}\| + \|\mathbf{U}\|_{\mathcal{L}} \|\mathbf{x}\| && \text{by Theorem 8.16} \\
&= \left(\|\mathbf{T}\|_{\mathcal{L}} + \|\mathbf{U}\|_{\mathcal{L}}\right) \|\mathbf{x}\| .
\end{aligned}$$
Therefore
$$\left(\|\mathbf{T}\|_{\mathcal{L}} + \|\mathbf{U}\|_{\mathcal{L}}\right) \in \left\{C > 0 \,\middle|\, \|(\mathbf{T} + \mathbf{U})(\mathbf{x})\| \leq C\|\mathbf{x}\| \text{ for all } \mathbf{x} \in \mathbb{R}^n\right\}$$
and hence
$$\left(\|\mathbf{T}\|_{\mathcal{L}} + \|\mathbf{U}\|_{\mathcal{L}}\right) \geq \inf \left\{C > 0 \,\middle|\, \|(\mathbf{T} + \mathbf{U})(\mathbf{x})\| \leq C\|\mathbf{x}\| \text{ for all } \mathbf{x} \in \mathbb{R}^n\right\} .$$
But by problem 1(c) we have that
$$\|\mathbf{T} + \mathbf{U}\|_{\mathcal{L}} = \inf \left\{C > 0 \,\middle|\, \|(\mathbf{T} + \mathbf{U})(\mathbf{x})\| \leq C\|\mathbf{x}\| \text{ for all } \mathbf{x} \in \mathbb{R}^n\right\} .$$
Combining the previous two lines gives
$$\|\mathbf{T} + \mathbf{U}\|_{\mathcal{L}} \leq \|\mathbf{T}\|_{\mathcal{L}} + \|\mathbf{U}\|_{\mathcal{L}}$$
as claimed. $\square$

(b) Prove that for any $c \in \mathbb{R}$,
$$\|c\mathbf{T}\|_{\mathcal{L}} = |c| \, \|\mathbf{T}\|_{\mathcal{L}}$$
where $c\mathbf{T}$ is the linear transformation defined by $(c\mathbf{T})(\mathbf{x}) = c\mathbf{T}(\mathbf{x})$.

Proof. We have
$$\begin{aligned}
\|c\mathbf{T}\|_{\mathcal{L}} &= \sup \left\{ \tfrac{\|(c\mathbf{T})(\mathbf{x})\|}{\|\mathbf{x}\|} \,\middle|\, \mathbf{x} \neq \mathbf{0} \right\} && \text{definition of operator norm} \\
&= \sup \left\{ \tfrac{\|c\mathbf{T}(\mathbf{x})\|}{\|\mathbf{x}\|} \,\middle|\, \mathbf{x} \neq \mathbf{0} \right\} && \text{definition of } c\mathbf{T} \\
&= \sup \left\{ |c| \tfrac{\|\mathbf{T}(\mathbf{x})\|}{\|\mathbf{x}\|} \,\middle|\, \mathbf{x} \neq \mathbf{0} \right\} && \text{properties of the norm on } \mathbb{R}^n \\
&= \sup \left(|c| \left\{ \tfrac{\|\mathbf{T}(\mathbf{x})\|}{\|\mathbf{x}\|} \,\middle|\, \mathbf{x} \neq \mathbf{0} \right\} \right) && \text{definition of } cA \text{ for } A \text{ a set and } c \in \mathbb{R} \\
&= |c| \sup \left\{ \tfrac{\|\mathbf{T}(\mathbf{x})\|}{\|\mathbf{x}\|} \,\middle|\, \mathbf{x} \neq \mathbf{0} \right\} && \text{properties of suprema} \\
&= |c| \, \|\mathbf{T}\|_{\mathcal{L}} && \text{definition of operator norm.}
\end{aligned}$$
Therefore $\|c\mathbf{T}\|_{\mathcal{L}} = |c| \, \|\mathbf{T}\|_{\mathcal{L}}$ as claimed. $\square$

[2] Note that one of the practical consequences of Problem (1) is that there are multiple equivalent definitions of the operator norm. That means each part of this problem could be done in different ways depending on what characterization of the operator norm that one chooses to work with.

(c) Prove that

$$\|\mathbf{T}\|_{\mathcal{L}} = 0$$

if and only if $\mathbf{T}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Proof. First assume that $\|\mathbf{T}\|_{\mathcal{L}} = 0$. Then by Theorem 8.16, we have for any $\mathbf{x} \in \mathbb{R}^n$ that

$$0 \leq \|\mathbf{T}(\mathbf{x})\| \leq \|\mathbf{T}\|_{\mathcal{L}} \|\mathbf{x}\| = 0$$

which implies that $\|\mathbf{T}(\mathbf{x})\| = 0$, and hence $\mathbf{T}(\mathbf{x}) = \mathbf{0}$. Thus $\|\mathbf{T}\|_{\mathcal{L}} = 0$ implies that $\mathbf{T}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Now assume that $\|\mathbf{T}(\mathbf{x})\| = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then for any $C > 0$ and $\mathbf{x} \in \mathbb{R}^n$, we have that

$$\|\mathbf{T}(\mathbf{x})\| = \|\mathbf{0}\| = 0 \leq C \|\mathbf{x}\| .$$

Thus

$$\left\{C > 0 \,\middle|\, \|\mathbf{T}(\mathbf{x})\| \leq C\|\mathbf{x}\| \text{ for all } \mathbf{x} \in \mathbb{R}^n\right\} = (0, +\infty)$$

and using problem 1(c) we conclude that

$$\|\mathbf{T}\|_{\mathcal{L}} = \inf\left\{C > 0 \,\middle|\, \|\mathbf{T}(\mathbf{x})\| \leq C\|\mathbf{x}\| \text{ for all } \mathbf{x} \in \mathbb{R}^n\right\} = \inf(0, +\infty) = 0.$$

Thus if $\mathbf{T}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$ then $\|\mathbf{T}\|_{\mathcal{L}} = 0$. $\square$

(3) (8.3.2) Let $\mathbf{a} \in \mathbb{R}^n$, and let $s, r \in \mathbb{R}$ satisfy $0 \le s < r$. Define

$$V = \{\mathbf{x} \in \mathbb{R}^n \mid s < \|\mathbf{x} - \mathbf{a}\| < r\} \quad \text{and} \quad E = \{\mathbf{x} \in \mathbb{R}^n \mid s \le \|\mathbf{x} - \mathbf{a}\| \le r\}.$$

Prove that V is open and that E is closed.

Proof 1: Using Theorem 8.24. We have that

$$\begin{aligned} V &= \{\mathbf{x} \in \mathbb{R}^n \mid s < \|\mathbf{x} - \mathbf{a}\| < r\} \\ &= \{\mathbf{x} \in \mathbb{R}^n \mid s < \|\mathbf{x} - \mathbf{a}\| \text{ and } \|\mathbf{x} - \mathbf{a}\| < r\} \\ &= \{\mathbf{x} \in \mathbb{R}^n \mid s < \|\mathbf{x} - \mathbf{a}\|\} \cap \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{a}\| < r\} \\ &= \bar{B}_s(\mathbf{a})^c \cap B_r(\mathbf{a}). \end{aligned}$$

The set $B_r(\mathbf{a})$ is open since open balls are open sets. Since closed balls are closed sets, the complement $\bar{B}_s(\mathbf{a})^c$ is open, so we have written V as the intersection of two open sets. Thus V is open by Theorem 8.24(ii).

Similarly, we have that

$$\begin{aligned} E &= \{\mathbf{x} \in \mathbb{R}^n \mid s \le \|\mathbf{x} - \mathbf{a}\| \le r\} \\ &= \{\mathbf{x} \in \mathbb{R}^n \mid s \le \|\mathbf{x} - \mathbf{a}\| \text{ and } \|\mathbf{x} - \mathbf{a}\| \le r\} \\ &= \{\mathbf{x} \in \mathbb{R}^n \mid s \le \|\mathbf{x} - \mathbf{a}\|\} \cap \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{a}\| \le r\} \\ &= B_s(\mathbf{a})^c \cap \bar{B}_r(\mathbf{a}). \end{aligned}$$

Since $B_s(\mathbf{a})$ is an open set[3] the complement $B_s(\mathbf{a})^c$ is a closed set. We have thus written E as the intersection of the closed sets $B_s(\mathbf{a})^c$ and $\bar{B}_r(\mathbf{a})$ so E is closed by Theorem 8.24(iv). □

Proof 2: Arguing directly from the definitions. Let $\mathbf{x} \in V$, i.e. assume that $s < \|\mathbf{x} - \mathbf{a}\| < r$. Let $\varepsilon = \min\{\|\mathbf{x} - \mathbf{a}\| - s, r - \|\mathbf{x} - \mathbf{a}\|\} > 0$. Then if $\mathbf{y} \in B_\varepsilon(\mathbf{x})$ we will have that

$$\begin{aligned} \|\mathbf{y} - \mathbf{a}\| &\le \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{a}\| && \text{by the triangle inequality} \\ &< \varepsilon + \|\mathbf{x} - \mathbf{a}\| && \text{since } \mathbf{y} \in B_\varepsilon(\mathbf{x}) \\ &\le (r - \|\mathbf{x} - \mathbf{a}\|) + \|\mathbf{x} - \mathbf{a}\| && \text{since } \varepsilon \le r - \|\mathbf{x} - \mathbf{a}\| \\ &= r. \end{aligned}$$

We also have that

$$\begin{aligned} \|\mathbf{y} - \mathbf{a}\| &\ge |\|\mathbf{y} - \mathbf{x}\| - \|\mathbf{x} - \mathbf{a}\|| && \text{by the triangle inequality} \\ &\ge \|\mathbf{x} - \mathbf{a}\| - \|\mathbf{y} - \mathbf{x}\| && \text{since } |c| \ge -c \text{ for all } c \in \mathbb{R} \\ &> \|\mathbf{x} - \mathbf{a}\| - \varepsilon && \text{since } \mathbf{y} \in B_\varepsilon(\mathbf{x}) \\ &\ge \|\mathbf{x} - \mathbf{a}\| - (\|\mathbf{x} - \mathbf{a}\| - s) && \text{since } \varepsilon \le \|\mathbf{x} - \mathbf{a}\| - s \\ &= s. \end{aligned}$$

Therefore, for $\mathbf{y} \in B_\varepsilon(\mathbf{x})$ we have that $s < \|\mathbf{y} - \mathbf{a}\| < r$ so $B_\varepsilon(\mathbf{x}) \subset V$. This shows that V is open.

To show that E is closed, we need to show that E^c is open. Let $\mathbf{x} \in E^c$. Then either $\|\mathbf{x} - \mathbf{a}\| > r$ or $\|\mathbf{x} - \mathbf{a}\| < s$. We first assume that $\|\mathbf{x} - \mathbf{a}\| > r$ and define $\varepsilon = \|\mathbf{x} - \mathbf{a}\| - r > 0$. Then if $\mathbf{y} \in B_\varepsilon(\mathbf{x})$ we will have that

$$\begin{aligned} \|\mathbf{y} - \mathbf{a}\| &\ge |\|\mathbf{y} - \mathbf{x}\| - \|\mathbf{x} - \mathbf{a}\|| && \text{by the triangle inequality} \\ &\ge \|\mathbf{x} - \mathbf{a}\| - \|\mathbf{y} - \mathbf{x}\| && \text{since } |c| \ge -c \text{ for all } c \in \mathbb{R} \\ &> \|\mathbf{x} - \mathbf{a}\| - \varepsilon && \text{since } \mathbf{y} \in B_\varepsilon(\mathbf{x}) \\ &= \|\mathbf{x} - \mathbf{a}\| - (\|\mathbf{x} - \mathbf{a}\| - r) && \text{since } \varepsilon = \|\mathbf{x} - \mathbf{a}\| - r \\ &= r. \end{aligned}$$

[3] If $s = 0$ we define $B_s(\mathbf{a}) = \emptyset$.

This shows that $B_\varepsilon(\mathbf{x}) \subset E^c$. We next assume that $\|\mathbf{x}-\mathbf{a}\| < s$, and choose $\varepsilon = s - \|\mathbf{x}-\mathbf{a}\| > 0$. Then if $\mathbf{y} \in B_\varepsilon(\mathbf{x})$ we will have that

$$\begin{aligned} \|\mathbf{y}-\mathbf{a}\| &\le \|\mathbf{y}-\mathbf{x}\| + \|\mathbf{x}-\mathbf{a}\| && \text{by the triangle inequality} \\ &< \varepsilon + \|\mathbf{x}-\mathbf{a}\| && \text{since } \mathbf{y} \in B_\varepsilon(\mathbf{x}) \\ &= (s - \|\mathbf{x}-\mathbf{a}\|) + \|\mathbf{x}-\mathbf{a}\| && \text{since } \varepsilon = s - \|\mathbf{x}-\mathbf{a}\| \\ &= s. \end{aligned}$$

Thus $B_\varepsilon(\mathbf{x}) \subset E^c$ in this case as well. We have thus shown that given any $\mathbf{x} \in E^c$ we can find an $\varepsilon > 0$ so that $B_\varepsilon(\mathbf{x}) \subset E^c$. Therefore E^c is open, and E is closed.

$\square$

(4) (8.3.9) Show that if $E \subset \mathbb{R}^n$ is a closed set and $\mathbf{a} \notin E$, then

$$\inf_{\mathbf{x} \in E} \|\mathbf{x} - \mathbf{a}\| > 0.$$

Proof. The assumption $\mathbf{a} \notin E$ is equivalent to $\mathbf{a} \in E^c$. Since E is closed, E^c is open, so there exists an $\varepsilon > 0$ so that $B_\varepsilon(\mathbf{a}) \subset E^c$. Therefore, given $\mathbf{x} \in E$, $\mathbf{x} \notin E^c$, and consequently $\mathbf{x} \notin B_\varepsilon(\mathbf{a})$ (or else we would have $\mathbf{x} \in B_\varepsilon(\mathbf{a}) \subset E^c$). But $\mathbf{x} \notin B_\varepsilon(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{a}\| < \varepsilon\}$ implies that

$$\|\mathbf{x} - \mathbf{a}\| \geq \varepsilon \text{ for all } \mathbf{x} \in E$$

and we can conclude that

$$\inf_{\mathbf{x} \in E} \|\mathbf{x} - \mathbf{a}\| \geq \varepsilon > 0.$$

□