Math 421, Homework #4
Due: Thursday, February 25, 5pm

Homework assignments should be submitted to my mailbox by 5pm Thursday. Please consult the homework rules and guidelines before completing the assignment. In particular:

- Write in compete sentences, organized into paragraphs.
- Leave plenty of room in between problems.
- Write only on the front side of each sheet of paper.
- Staple!
- Write on your assignment the names of any persons or sources consulted during its completion (other than the course text or instructor).

(1) (8.2.11) Let $\mathbf{T} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, and define

$$M_1 := \sup_{\|\mathbf{x}\|=1} \|\mathbf{T}(\mathbf{x})\| = \sup\left\{\|\mathbf{T}(\mathbf{x})\| \,\middle|\, \mathbf{x} \in \mathbb{R}^n; \|\mathbf{x}\| = 1\right\}$$

$$M_2 := \inf\left\{C > 0 \,\middle|\, \|\mathbf{T}(\mathbf{x})\| \leq C\|\mathbf{x}\| \text{ for all } \mathbf{x} \in \mathbb{R}^n\right\}.$$

(a) Prove that $M_1 \leq \|\mathbf{T}\|$.

(b) Using the linear property of $\mathbf{T}$, prove that if $\mathbf{x} \neq \mathbf{0}$, then

$$\frac{\|\mathbf{T}(\mathbf{x})\|}{\|\mathbf{x}\|} \leq M_1.$$

(c) Prove that $M_1 = M_2 = \|\mathbf{T}\|$.

(2) Let $\mathbf{T}$, $\mathbf{U} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

(a) Prove that

$$\|\mathbf{T} + \mathbf{U}\| \leq \|\mathbf{T}\| + \|\mathbf{U}\|$$

where $\mathbf{T} + \mathbf{U}$ is the linear transformation defined by $(\mathbf{T} + \mathbf{U})(\mathbf{x}) = \mathbf{T}(\mathbf{x}) + \mathbf{U}(\mathbf{x})$.

(b) Prove that for any $c \in \mathbb{R}$,

$$\|c\mathbf{T}\| = |c|\|\mathbf{T}\|$$

where $c\mathbf{T}$ is the linear transformation defined by $(c\mathbf{T})(\mathbf{x}) = c\mathbf{T}(\mathbf{x})$.

(c) Prove that

$$\|\mathbf{T}\| = 0$$

if and only if $\mathbf{T}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$.

(3) (8.3.2) Let $\mathbf{a} \in \mathbb{R}^n$, and let s, $r \in \mathbb{R}$ satisfy $0 \leq s < r$. Define

$$V = \{\mathbf{x} \in \mathbb{R}^n \mid s < \|\mathbf{x} - \mathbf{a}\| < r\} \quad \text{and} \quad E = \{\mathbf{x} \in \mathbb{R}^n \mid s \leq \|\mathbf{x} - \mathbf{a}\| \leq r\}.$$

Prove that V is open and that E is closed.

(4) (8.3.9) Show that if $E \subset \mathbb{R}^n$ is a closed set and $\mathbf{a} \notin E$, then

$$\inf_{\mathbf{x} \in E} \|\mathbf{x} - \mathbf{a}\| > 0.$$