Math 421, Homework #3 Solutions

(1) Consider a function $f : \mathbb{R} \to \mathbb{R}$, and assume that f is continuous at $x_0 \in \mathbb{R}$ and locally integrable on $\mathbb{R}$. Prove that

$$\lim_{\delta \to 0+} \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx = f(x_0).$$

Proof 1. In order to prove that

$$\lim_{\delta \to 0+} \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx = f(x_0).$$

we need to show that for any $\varepsilon > 0$, there exists a $\delta' > 0$ so that

$$\delta \in (0, \delta') \qquad \Rightarrow \qquad \left| \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx - f(x_0) \right| < \varepsilon.$$

Let $\varepsilon > 0$. Since f is assumed continuous at x_0, we can find a $\delta' > 0$ so that

$$|x - x_0| < \delta' \qquad \Rightarrow \qquad |f(x) - f(x_0)| < \varepsilon.$$

For $\delta \in (0, \delta')$, we therefore have that

$$\begin{aligned}
\left| \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx - f(x_0) \right| &= \left| \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx - f(x_0) \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} 1\,dx \right| \\
&= \left| \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx - \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x_0)\,dx \right| \\
&= \left| \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x) - f(x_0)\,dx \right| \\
&\leq \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} |f(x) - f(x_0)|\,dx \\
&< \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} \varepsilon\,dx && \text{since } |x - x_0| < \delta < \delta' \\
&= \varepsilon.
\end{aligned}$$

Hence $\delta \in (0, \delta')$ implies that

$$\left| \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx - f(x_0) \right| < \varepsilon$$

as required. □

Proof 2. Here we will use the Fundamental Theorem of Calculus (FTC).

Define

$$F(x) = \int_c^x f(t)\,dt$$

for any $c \in \mathbb{R}$. Note that since f is assumed to be continuous at x_0, the FTC implies that F is differentiable at x_0, and that $F'(x_0) = f(x_0)$.[1]

We then have that

[1] Strictly speaking, the statement of the FTC assumed that $c < x$ in the definition $\int_c^x f(t)\,dt$, but this assumption is easily removed since for any other $c' \in \mathbb{R}$ we have that

$$\int_{c'}^x f(t)\,dt = \int_c^{c'} f(t)\,dt + \int_c^x f(t)\,dt$$

so choosing a different c in the definition of F merely changes the function by a constant.

$$
\begin{aligned}
\lim_{\delta \to 0^+} \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx &= \lim_{\delta \to 0^+} \frac{1}{2\delta} \left(\int_c^{x_0+\delta} f(x)\,dx - \int_c^{x_0-\delta} f(x)\,dx \right) \\
&= \lim_{\delta \to 0^+} \frac{1}{2\delta} \left(F(x_0+\delta) - F(x_0-\delta) \right) \\
&= \frac{1}{2} \left(\lim_{\delta \to 0^+} \frac{F(x_0+\delta) - F(x_0)}{\delta} + \lim_{\delta \to 0^+} \frac{F(x_0-\delta) - F(x_0)}{-\delta} \right) \\
&= \frac{1}{2} \left(F'(x_0) + F'(x_0) \right) \\
&= F'(x_0) = f(x_0).
\end{aligned}
$$

$\square$

(2) (5.3.9) Suppose that $f : [a, b] \to \mathbb{R}$ is continuously differentiable and 1–1 on $[a, b]$. Prove that

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(x)\,dx = bf(b) - af(a). \tag{1}$$

Proof. Since f is continuous and 1–1, the inverse f^{-1} of f is defined and continuous. Moreover, since f is assumed to be continuously differentiable, we can apply Theorem 5.35 (with f^{-1} here playing the role of f in the theorem, and f here playing the role of ϕ in the theorem) that

$$\begin{aligned}\int_{f(a)}^{f(b)} f^{-1}(x)\,dx &= \int_a^b f^{-1}(f(x))f'(x)\,dx \\ &= \int_a^b xf'(x)\,dx.\end{aligned}$$

Integrating this last expression by parts, we continue

$$\begin{aligned}&= xf(x)|_a^b - \int_a^b \tfrac{d}{dx}(x)f(x)\,dx \\ &= bf(b) - af(a) - \int_a^b f(x)\,dx.\end{aligned}$$

We have thus shown that

$$\int_{f(a)}^{f(b)} f^{-1}(x)\,dx = bf(b) - af(a) - \int_a^b f(x)\,dx$$

which is equivalent to (1) above. □

(3) (cf. 5.4.6) Let $f,\ g : [0, \infty) \to \mathbb{R}$ be locally integrable functions, and assume that $g(x) \geq 0$ for all $x \in [0, \infty)$. Assume that the limit

$$L := \lim_{x \to \infty} \frac{f(x)}{g(x)}$$

exists and satisfies $L \in (0, \infty)$. Prove that f is improperly integrable on $[0, \infty)$ if and only if g is improperly integrable on $[0, \infty)$.

Proof. According to the definition of limit, the equation

$$L = \lim_{x \to \infty} \frac{f(x)}{g(x)}$$

means that for any $\varepsilon > 0$, there exists an N, so that $x > N$ implies that

$$\left| \frac{f(x)}{g(x)} - L \right| < \varepsilon.$$

Since $L > 0$ is finite, we can, in particular, find an N, so that $x > N$ implies that

$$\left| \frac{f(x)}{g(x)} - L \right| < \frac{L}{2},$$

or equivalently, so that

$$0 < \frac{L}{2} < \frac{f(x)}{g(x)} < \frac{3L}{2}$$

for all $x > N$. Note that this implies that $g(x)$ is nonzero for all $x > N$, and since $g(x) \geq 0$ for all $x \geq 0$, we have that $g(x) > 0$ for all $x > N$. We can therefore multiply this inequality through by $g(x)$ to find that

$$0 < \frac{L}{2} g(x) < f(x) < \frac{3L}{2} g(x)$$

for all $x > N$.

Now, if g is improperly integrable on $[0, \infty)$, then $\frac{3L}{2} g$ is improperly integrable on $[0, \infty)$ (Theorem 5.42), and therefore $\frac{3L}{2} g$ is improperly integrable on $[N, \infty)$. The inequality

$$0 < f(x) < \frac{3L}{2} g(x) \qquad \text{for all } x > N$$

then allows us to apply the comparison theorem for improper integrals (Theorem 5.43) to conclude that f is improperly integrable on $[N, \infty)$. Since f is assumed locally integrable on $[0, \infty)$, f being improperly integrable on $[N, \infty)$ is equivalent to f being improperly integrable on $[0, \infty)$.

Similarly, if f is improperly integrable on $[0, \infty)$, we use the inequality

$$0 < \frac{L}{2} g(x) < f(x) \qquad \text{for all } x > N$$

with the comparison theorem for improper integrals to conclude that $\frac{L}{2} g$ and g are improperly integrable on $[N, \infty)$, which, with the assumption of local integrability of g, implies that g is improperly integrable on $[0, \infty)$.

$\square$

(4) Let $f : [0, +\infty) \to \mathbb{R}$ be a locally integrable function. Show that f is improperly integrable on $[0, +\infty)$ if and only if for every $\varepsilon > 0$ there exists an $R > 0$ so that if $y > x > R$ then $\left|\int_x^y f(t)\,dt\right| < \varepsilon$.

Proof. We first assume that f is improperly integrable on $[0, \infty)$. This means that the limit

$$L := \lim_{r\to\infty} \int_0^r f(t)\,dt$$

exists (and is finite). By definition, this means that for any $\varepsilon' > 0$ we can find an $R \in \mathbb{R}$ so that for $r > R$ we have that

$$\left|\int_0^r f(t)\,dt - L\right| < \varepsilon'.$$

Given, $\varepsilon > 0$, we can thus find an $R > 0$ so that

$$\left|\int_0^r f(t)\,dt - L\right| < \frac{\varepsilon}{2} \text{ for } r > R \tag{2}$$

(i.e. we apply the previous definition with $\varepsilon' = \varepsilon/2$). We then find that if $y > x > R$ that

$$\begin{aligned}
\left|\int_x^y f(t)\,dt\right| &= \left|\int_0^y f(t)\,dt - \int_0^x f(t)\,dt\right| && \text{by Theorem 5.20}\\
&= \left|\left(\int_0^y f(t)\,dt - L\right) + \left(L - \int_0^x f(t)\,dt\right)\right|\\
&\le \left|\int_0^y f(t)\,dt - L\right| + \left|L - \int_0^x f(t)\,dt\right| && \text{by the triangle inequality}\\
&= \left|\int_0^y f(t)\,dt - L\right| + \left|\int_0^x f(t)\,dt - L\right|\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon && \text{by (2), since } x > R \text{ and } y > R.
\end{aligned}$$

Thus, improper integrability of f on $[0, \infty)$ implies that for any $\varepsilon > 0$ there is an $R > 0$ so that $\left|\int_x^y f(t)\,dt\right| < \varepsilon$ provided $y > x > R$.

We next assume that for any $\varepsilon' > 0$ there exists an $R > 0$ so that for $y > x > R$, $\left|\int_x^y f(t)\,dt\right| < \varepsilon'$. Define a sequence $x_n = \int_0^n f(t)\,dt$. Then, given $\varepsilon > 0$, we can choose an $R > 0$ so that $\left|\int_x^y f(t)\,dt\right| < \varepsilon$, and choosing an $N \in \mathbb{N}$ with $N \ge R$ we see that for $n > m > N$ that

$$\begin{aligned}
|x_n - x_m| &= \left|\int_0^n f(t)\,dt - \int_0^m f(t)\,dt\right| && \text{by definition of } x_j\\
&= \left|\int_m^n f(t)\,dt\right| && \text{by Theorem 5.20}\\
&< \varepsilon.
\end{aligned}$$

Therefore x_n is a Cauchy sequence and hence has a limit. Let $L = \lim_{n\to\infty} x_n$. We claim that $\lim_{r\to\infty} \int_0^r f(t)\,dt = L$. To prove this we need to demonstrate that for any $\varepsilon > 0$, there is an $R > 0$ so that if $r > R$ then $\left|\int_0^r f(t)\,dt - L\right| < \varepsilon$. Let $\varepsilon > 0$. Choose an R' so that

$$\left|\int_x^y f(t)\,dt\right| < \frac{\varepsilon}{2} \text{ if } y > x > R', \tag{3}$$

which we can do by assumption, and choose an $N \in \mathbb{N}$ so that

$$|x_n - L| < \frac{\varepsilon}{2} \text{ if } n > N, \tag{4}$$

which we can do by the fact that $\lim_{n\to\infty} x_n = L$. Then defining $R = \max\{R', N\}$ we choose a natural number $n > R$ and find that if $r > R$ then

$$\begin{aligned}
\left|\int_0^r f(t)\,dt - L\right| &= \left|\left(\int_0^r f(t)\,dt - x_n\right) + (x_n - L)\right| \\
&\le \left|\int_0^r f(t)\,dt - x_n\right| + |x_n - L| && \text{by the triangle inequality} \\
&< \left|\int_0^r f(t)\,dt - x_n\right| + \frac{\varepsilon}{2} && \text{by (4) since } n > R \ge N \\
&= \left|\int_0^r f(t)\,dt - \int_0^n f(t)\,dt\right| + \frac{\varepsilon}{2} && \text{by definition of } x_n \\
&= \left|\int_n^r f(t)\,dt\right| + \frac{\varepsilon}{2} && \text{Theorem 5.20} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon && \text{by (3) since } r > R \ge R' \text{ and } n > R \ge R'.
\end{aligned}$$

We conclude that $\lim_{r\to\infty} \int_0^r f(t)\,dt = L$ and thus f is improperly integrable on $[0,\infty)$. □

(5) (5.4.7)

(a) Suppose that f is improperly integrable on $[0, +\infty)$. Prove that if $\lim_{x\to\infty} f(x)$ exists, then $\lim_{x\to\infty} f(x) = 0$.

Proof. Let
$$L = \lim_{x\to\infty} f(x).$$
We will argue by contradiction to show that $L = 0$. Assume initially that $L > 0$. Then according to the definition of $L = \lim_{x\to\infty} f(x)$ there is an $R_1 \geq 0$ so that
$$|f(x) - L| < \frac{L}{2} \text{ if } x > R_1$$
from which we can conclude that $-L/2 < f(x) - L < L/2$ and thus
$$f(x) > \frac{L}{2} > 0 \text{ if } x > R_1. \tag{5}$$
We claim that this implies that f is not improperly integrable on $[0, \infty)$. If f were improperly integrable on $[0, \infty)$ then problem (4) would allow us to find an $R_2 > 0$ so that
$$\left| \int_x^y f(t)\, dt \right| < 1 \text{ for all } y > x > R_2. \tag{6}$$
Choosing an $x > \max\{R_1, R_2\}$ and defining $y = x + 3/L$, we conclude
$$\begin{aligned} \int_x^y f(t)\, dt &\geq \int_x^y \frac{L}{2}\, dt && \text{by the comparison theorem and (5)} \\ &= (y - x)\frac{L}{2} \\ &= \frac{3}{L}\frac{L}{2} = \frac{3}{2} > 1 \end{aligned}$$
which contradict (6). Thus f is not improperly integrable on $[0, \infty)$ if $L > 0$.

If $L < 0$ we can apply the argument of the previous paragraph to $-f$ (which will limit to $-L > 0$) to conclude that $-f$ is not improperly integrable on $[0, \infty)$ and thus f is not integrable on $[0, \infty)$. We conclude that $L = 0$. □

(b) Let

$$f(x) = \begin{cases} 1 & \text{if } x \in [n, n+2^{-n}) \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is improperly integrable on $[0, +\infty)$ but $\lim_{x\to\infty} f(x)$ does not exist. (Note that this example shows that we can't eliminate the assumption that the limit $\lim_{x\to\infty} f(x)$ exists in part (a).)

Proof. Let $n \in \mathbb{N}$. We have that

$$\int_n^{n+1} f(x)\,dx = \int_n^{n+2^{-n}} f(x)\,dx + \int_{n+2^{-n}}^{n+1} f(x)\,dx.$$

Since $f(x) = 1$ for $x \in [n, n+2^{-n}] \setminus \{n+2^{-n}\}$ it follows from problem #2 on homework #2 that $\int_n^{n+2^{-n}} f(x)\,dx = \int_n^{n+2^{-n}} 1\,dx = 2^{-n}$. Similarly, since $f(x) = 0$ for $x \in [n+2^{-n}, n+1] \setminus \{n+1\}$, the second integral on the right vanishes. Therefore

$$\int_n^{n+1} f(x)\,dx = 2^{-n} \text{ for } n \in \mathbb{N}. \tag{7}$$

Similarly, since $f(x) = 0$ for $x \in [0,1)$, we have that $\int_0^1 f(x)\,dx = 0$.[2]

To see that f is improperly integrable, we first observe that f is locally integrable because on any closed interval contained in $[0,\infty)$, f has a finite number of discontinuities (problem #1 from homework #2). Defining $F(x) = \int_0^x f(t)\,dt$, we observe that since f is nonnegative, we can use the comparison theorem to conclude that for $y > x$,

$$F(y) - F(x) = \int_0^y f(t)\,dt - \int_0^x f(t)\,dt = \int_x^y f(t)\,dt \geq 0.$$

Therefore F is an increasing function. To show that $\lim_{x\to\infty} F(x)$ exists (and hence that f is improperly integrable on $[0,\infty)$) if suffices to show that F is bounded above. Let $x \in [0,\infty)$. Choosing a natural number $N > x$, we have that

$$\begin{aligned}
F(x) &\leq F(N) && \text{since } F \text{ is increasing} \\
&= \int_0^N f(t)\,dt && \text{by definition of } F \\
&= \sum_{n=0}^{N} \int_n^{n+1} f(t)\,dt && \text{by Theorem} \\
&= \sum_{n=1}^{N} \int_n^{n+1} f(t)\,dt && \text{since } \int_0^1 f(x)\,dx = 0 \\
&= \sum_{n=1}^{N} 2^{-n} && \text{by (7)} \\
&= 1 - \frac{1}{2^N} && \text{by the geometric sum formula (see proof of Thm 6.7)} \\
&< 1.
\end{aligned}$$

Therefore $F(x) < 1$ for any $x \in [0,\infty)$, so F is bounded above, and as observed above, this implies that $\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \int_0^x f(t)\,dt$ exists. [3]

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[2] If one uses the convention that the natural numbers include zero, then we would have instead that $\int_0^1 f(x)\,dx = 1$ instead, but this ultimately has no effect on the improper integrability of f.

[3] With slightly more work, we can show that $\int_0^\infty f(t)\,dt = 1$. Indeed, given any $x \in [0,\infty)$ we can argue that if $n \in \mathbb{N}$ is less than x, then $\int_0^x f(t)\,dt \geq 1 - 2^{-n}$. Therefore for any $\varepsilon > 0$ we can find an $R > 0$ so that if $x > R$, $\int_0^x f(t)\,dt \in (1-\varepsilon, 1)$.

Finally, we show that $\lim_{x\to\infty} f(x)$ doesn't exist. Arguing by contradiction we assume that $L = \lim_{x\to\infty} f(x)$ exists. Then there exists an $R \in \mathbb{R}$ so that

$$|f(x) - L| < \frac{1}{2} \text{ if if } x > R,$$

or equivalently that

$$-\frac{1}{2} < f(x) - L < \frac{1}{2} \text{ if if } x > R. \tag{8}$$

Choosing a natural number $N > R$, we get that $f(N) = 1$ so we can use (8) to conclude that $L > \frac{1}{2}$. Meanwhile using that $N + \frac{1}{2} > N > R$ with $f(N + \frac{1}{2}) = 0$ and (8) we can also conclude that $L < \frac{1}{2}$ which is a contradiction. Therefore $\lim_{x\to\infty} f(x)$ doesn't exist. □