Math 421, Homework #3
Due: Wednesday, February 10

Homework assignments should be submitted during lecture. Please consult the homework rules and guidelines before completing the assignment. In particular:

- Write in compete sentences, organized into paragraphs.
- Leave plenty of room in between problems.
- Write only on the front side of each sheet of paper.
- Staple!
- Write on your assignment the names of any persons or sources consulted during its completion (other than the course text or instructor).

(1) Consider a function $f : \mathbb{R} \to \mathbb{R}$, and assume that f is continuous at $x_0 \in \mathbb{R}$ and locally integrable on $\mathbb{R}$. Prove that

$$\lim_{\delta \to 0^+} \frac{1}{2\delta} \int_{x_0 - \delta}^{x_0 + \delta} f(x)\, dx = f(x_0).$$

(2) (5.3.9) Suppose that $f : [a, b] \to \mathbb{R}$ is continuously differentiable and 1–1 on $[a, b]$. Prove that

$$\int_a^b f(x)\, dx + \int_{f(a)}^{f(b)} f^{-1}(x)\, dx = bf(b) - af(a).$$

(Hint: Try to evaluate the second integral above by using substitution and integration by parts. You may use the fact that a continuous, 1–1 function has a continuous inverse without proving this.)

(3) (cf. 5.4.6) Let $f,\ g : [0, \infty) \to \mathbb{R}$ be locally integrable functions, and assume that $g(x) \geq 0$ for all $x \in [0, \infty)$. Assume that the limit

$$L := \lim_{x \to \infty} \frac{f(x)}{g(x)}$$

exists and satisfies $L \in (0, \infty)$. Prove that f is improperly integrable on $[0, \infty)$ if and only if g is improperly integrable on $[0, \infty)$.

(4) Let $f : [0, +\infty) \to \mathbb{R}$ be a locally integrable function. Show that f is improperly integrable on $[0, +\infty)$ if and only if for every $\varepsilon > 0$ there exists an $R > 0$ so that if $y > x > R$ then $\left| \int_x^y f(t)\, dt \right| < \varepsilon$. (Hint: The difficult part of this proof is showing the second statement implies the first since to use the definition of limit, you need a proposed value for $\lim_{d \to \infty} \int_0^d f(x)\, dx$. One way to deal with this difficulty is to show that the second statement implies that the sequence $x_k := \int_0^k f(t)\, dt$ (with $k \in \mathbb{N}$) is a Cauchy sequence.)

(5) (5.4.7)

(a) Suppose that f is improperly integrable on $[0, +\infty)$. Prove that if $\lim_{x \to \infty} f(x)$ exists, then $\lim_{x \to \infty} f(x) = 0$.

(b) Let

$$f(x) = \begin{cases} 1 & \text{if } x \in [n, n + 2^{-n}) \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is improperly integrable on $[0, +\infty)$ but $\lim_{x \to \infty} f(x)$ does not exist. (Note that this example shows that we can't eliminate the assumption that the limit $\lim_{x \to \infty} f(x)$ exists in part (a).)