Math 421, Homework #2 Solutions

(1) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Assume that f has a finite number of discontinuities, i.e. assume there exists a finite subset E of $[a, b]$ so that f is continuous at all $x \in [a, b] \setminus E$. Prove that f is integrable on $[a, b]$.

Proof. Define the set $E_1 = E \cup \{a, b\}$, and label the elements of E_1 by

$$a = q_0 < q_1 < \cdots < q_n = b.$$

and define a quantity

$$\Delta q = \min_{j \in \{1,\ldots,n\}} q_j - q_{j-1}$$

so that Δq gives the smallest distance between successive q_j's. Since f is assumed to be bounded on $[a, b]$, there is an $M > 0$ so that

$$-M \leq f(x) \leq M \qquad \text{for all } x \in [a, b].$$

Let $\varepsilon > 0$, and define $\delta > 0$ by

$$\delta = \min\left\{\frac{\varepsilon}{8Mn}, \frac{\Delta q}{4}\right\}.$$

Define a partition $Q = \{x_0, \ldots, x_{2n+1}\}$ of $[a, b]$ by $x_0 = a$, $x_{2n+1} = b$,

$$x_{2j+1} = q_j + \delta \qquad \text{if } j \in \{0, \ldots, n-1\},$$

and

$$x_{2j} = q_j - \delta \qquad \text{if } j \in \{1, \ldots, n\}.$$

Since we chose $\delta \leq \Delta q/4$ it follows that $x_j > x_i$ for $j > i$. Note that by construction $q_j \in (x_{2j}, x_{2j+1})$ if $j \in \{1, \ldots, n-1\}$, while $q_0 \in [x_0, x_1)$ and $q_n \in (x_{2n}, x_{2n+1}]$. Hence f is continuous, and therefore integrable, on each of the intervals $[x_{2j-1}, x_{2j}]$ for $j \in \{1, \ldots, n\}$. Therefore we can choose a partition P_j of $[x_{2j-1}, x_{2j}]$ so that

$$U(f, P_j) - L(f, P_j) < \frac{\varepsilon}{2n}$$

for each $j \in \{1, \ldots, n\}$. Defining, $P = Q \cup \left(\cup_{j=1}^{n} P_j\right)$, we then have that

$$\begin{aligned}
U(f, P) - L(f, P) &= [\sup f([a, x_1]) - \inf f([a, x_1])](x_1 - a) \\
&\quad + [\sup f([x_{2n}, b]) - \inf f([x_{2n}, b])](b - x_{2n}) \\
&\quad + \sum_{j=1}^{n-1} [\sup f([x_{2j}, x_{2j+1}]) - \inf f([x_{2j}, x_{2j+1}])](x_{2j+1} - x_{2j}) \\
&\quad + \sum_{j=1}^{n} U(f, P_j) - L(f, P_j) \\
&< 2M\delta + 2M\delta + \sum_{j=1}^{n-1} 2M(2\delta) + \sum_{j=1}^{n} \frac{\varepsilon}{2n} \\
&= 4Mn\delta + \frac{\varepsilon}{2} \\
&\leq 4Mn\frac{\varepsilon}{8Mn} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

We therefore can conclude that f is integrable on $[a, b]$.

$\square$

(2) (a) Consider a function $f : [a, b] \to \mathbb{R}$, and assume that there is a single point at which f is nonzero, i.e. assume that there is a point $c \in [a, b]$ so that f satisfies $f(x) = 0$ for all $x \in [a, b] \setminus \{c\}$. Prove that f is integrable on $[a, b]$ and that $\int_a^b f(x)\,dx = 0$.

Proof. Since $f([a, b]) = \{0, f(c)\}$ by assumption, it follows that f is bounded on $[a, b]$. Furthermore, since $f(x) = 0$ for $x \in [a, b] \setminus \{c\}$, it follows that f is continuous on $[a, b] \setminus \{c\}$. Therefore by Problem (1), f is integrable on $[a, b]$.

To prove that $\int_a^b f(x)\,dx = 0$, we initially assume that $f(c) \geq 0$. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. We claim that $L(f, P) = 0$. Indeed, if $c \notin [x_{j-1}, x_j]$ then $f(x)$ is identically zero on $[x_{j-1}, x_j]$ so

$$m_j(f) := \inf f([x_{j-1}, x_j]) = \inf \{0\} = 0.$$

On the other hand if $c \in [x_{j-1}, x_j]$ then $f([x_{j-1}, x_j]) = \{0, f(c)\}$, and since we are assuming $f(c) \geq 0$, we again have that

$$m_j(f) = \inf f([x_{j-1}, x_j]) = \inf \{0, f(c)\} = 0.$$

Thus $L(f, P) = 0$ as claimed. Since we have already shown that f is integrable on $[a, b]$ we have

$$\begin{aligned}
\int_a^b f(x)\,dx &= (L)\int_a^b f(x)\,dx \\
&= \sup \{L(f, P) \mid P \text{ a partition of } [a, b]\} \\
&= \sup \{0\} \\
&= 0.
\end{aligned}$$

Finally, to deal with the case that $f(c) < 0$, we can apply the previous argument to the function $-f$ to conclude that $\int_a^b [-f(x)]\,dx = 0$ and thus $\int_a^b f(x)\,dx = 0$ as well. $\square$

(b) Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Let $g : [a, b] \to \mathbb{R}$ be a function which agrees with f at all points in $[a, b]$ except for one, i.e. assume there exists a $c \in [a, b]$ so that $g(x) = f(x)$ for all $x \in [a, b] \setminus \{c\}$. Prove that g is integrable on $[a, b]$ and that $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$.

Proof. Define $h(x) = f(x) - g(x)$. According to our assumptions $h(x) = 0$ on $[a, b] \setminus \{c\}$. Therefore part (a) implies that h is integrable on $[a, b]$ and that

$$\int_a^b h(x)\,dx = 0.$$

Then, since $g(x) = f(x) - h(x)$ it follows from Theorem 5.19 that g is integrable on $[a, b]$ and that

$$\begin{aligned}
\int_a^b g(x)\,dx &= \int_a^b f(x) - h(x)\,dx \\
&= \int_a^b f(x)\,dx - \int_a^b h(x)\,dx \\
&= \int_a^b f(x)\,dx.
\end{aligned}$$

$\square$

(c) (5.1.6) Let $f : [a, b] \to \mathbb{R}$ be an integrable function, and assume that $g : [a, b] \to \mathbb{R}$ agrees with f except on a finite set, i.e. assume there exists a finite set E so that $g(x) = f(x)$ for all $x \in [a, b] \setminus E$. Prove that g is integrable on $[a, b]$ and that $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$.

Proof. Assume that $E = \{x_1, \dots, x_n\}$. For $k \in \{1, \dots, n-1\}$, define functions g_k on $[a, b]$ by

$$g_k(x) = \begin{cases} f(x) & \text{if } x \in [a, b] \setminus \{x_1, \dots, x_k\} \\ g(x) & \text{if } x \in \{x_{k+1}, \dots, x_n\}, \end{cases}$$

and define $g_0 = f$ and $g_n = g$. Then, it follows immediately from this definition that

$$g_k(x) = g_{k-1}(x) \text{ for all } x \in [a, b] \setminus \{x_k\}.$$

We will argue by induction to prove that for all $k \in \{0, 1, \dots, n\}$ that g_k is integrable on $[a, b]$ and that

$$\int_a^b g_k(x)\,dx = \int_a^b f(x)\,dx. \tag{1}$$

Since $g = g_n$, this will prove the desired result. As our base case, we take $k = 0$, in which case (1) holds trivially. For our inductive step, we assume that for some $k \geq 0$ that g_k is integrable on $[a, b]$ and that (1) holds. Then, since g_{k+1} agrees with g_k except for at x_{k+1}, we can use part (b) to conclude that g_{k+1} is integrable on $[a, b]$ and that

$$\int_a^b g_{k+1}(x)\,dx = \int_a^b g_k(x)\,dx = \int_a^b f(x)\,dx.$$

□

(3) (5.2.5) Prove that if f is integrable on $[0,1]$ and $\beta > 0$, then

$$\lim_{n\to\infty} n^{\alpha} \int_0^{1/n^{\beta}} f(x)\,dx = 0$$

for all $\alpha < \beta$.

Proof. Since f is assumed integrable on $[a,b]$, f must be bounded, i.e. there exists an $M > 0$ so that

$$|f(x)| \le M \text{ for all } x \in [a,b].$$

Using Theorem 5.22, and the comparison theorem (Theorem 5.21), we can conclude for $n > 0$ that

$$\begin{aligned} \left| n^{\alpha} \int_0^{1/n^{\beta}} f(x)\,dx \right| &\le n^{\alpha} \int_0^{1/n^{\beta}} |f(x)|\,dx \\ &\le n^{\alpha} \int_0^{1/n^{\beta}} M\,dx \\ &= Mn^{\alpha}(1/n^{\beta} - 0) \\ &= Mn^{\alpha-\beta}, \end{aligned}$$

so

$$0 \le \left| n^{\alpha} \int_0^{1/n^{\beta}} f(x)\,dx \right| \le Mn^{\alpha-\beta}. \tag{2}$$

Since we assume $\beta > \alpha$, we know that $n^{\alpha-\beta} \to 0$ as $n \to \infty$, it follows from (2) and the squeeze theorem that

$$\lim_{n\to\infty} n^{\alpha} \int_0^{1/n^{\beta}} f(x)\,dx = 0.$$

□

(4) (5.2.8) Let f be continuous on a closed, nondegenerate interval$[a,b]$, let

$$M = \sup_{x\in[a,b]} |f(x)|,$$

and assume that $M > 0$.

(a) Prove that if $p > 0$, then for every $\varepsilon \in (0, M)$ there is a nondegenerate interval $I_\varepsilon \subset [a,b]$ such that

$$(M-\varepsilon)^p |I_\varepsilon| \le \int_a^b |f(x)|^p \le M^p(b-a) \tag{3}$$

where $|I_\varepsilon|$ denotes the length of the interval I_ε.

Proof. By definition of supremum, we have that

$$|f(x)| \le \sup_{x\in[a,b]} |f(x)| = M \qquad \text{for all } x \in [a,b],$$

and since x^p is an increasing function on $[0,\infty)$, we can raise each term to the p power to conclude that

$$|f(x)|^p \le M^p \qquad \text{for all } x \in [a,b].$$

Integrating on $[a,b]$ and using the comparison theorem (Theorem 5.21), we conclude that

$$\int_a^b |f(x)|^p\,dx \le \int_a^b M^p\,dx = M^p(b-a). \tag{4}$$

Since f is assumed to be continuous, $|f|$ is also continuous, so the extreme value theorem let's us conclude that there is an $x_0 \in [a,b]$ satisfying $|f(x_0)| = \sup_{x\in[a,b]} |f(x)| = M$. Let $\varepsilon > 0$. Again using continuity of $|f|$, there exists a $\delta > 0$ so that for $x \in [a,b]$ satisfying $|x - x_0| < \delta$, we will have that $||f(x)| - M| = ||f(x)| - |f(x_0)|| < \varepsilon$. If we choose a closed nondegenerate interval[1] $I = [c,d] \subset (x_0 - \delta, x_0 + \delta) \cap [a,b]$ then

$$||f(x)| - M| < \varepsilon \qquad \text{for all } x \in I$$

which implies that $f(x) \ge M - \varepsilon \ge 0$ for all $x \in I$. Again using that x^p in increasing on $[0,\infty)$ we can conclude that

$$|f(x)|^p > (M-\varepsilon)^p \qquad \text{for all } x \in I$$

Using the comparison theorem, it then follows that

$$\int_c^d |f(x)|^p\,dx \ge \int_c^d (M-\varepsilon)^p\,dx = (M-\varepsilon)^p(d-c) = (M-\varepsilon)^p\,|I|\,. \tag{5}$$

Moreover, using Theorem 5.20 we have that

$$\begin{aligned}\int_a^b |f(x)|^p\,dx &= \int_a^c |f(x)|^p\,dx + \int_c^d |f(x)|^p\,dx + \int_d^b |f(x)|^p\,dx \\ &\ge \int_c^d |f(x)|^p\,dx.\end{aligned}$$

Here we've used the comparison theorem with $|f(x)|^p \ge 0$ to conclude that $\int_a^c |f(x)|^p\,dx \ge 0$ and $\int_d^b |f(x)|^p\,dx \ge 0$. This with (5) yields

$$\int_a^b |f(x)|^p\,dx \ge (M-\varepsilon)^p\,|I|\,. \tag{6}$$

Together (4) and (6) yield (3). ☐

[1] If $x_0 \in (a,b)$ we can choose $I = [x_0 - \delta', x_0 + \delta']$ where $\delta' > 0$ is $\delta' = \min\{\delta/2, b - x_0, x_0 - a\}$. If $x_0 = a$ we can choose $I = [a, a+\delta']$ where $\delta' = \min\{\delta/2, b-a\} > 0$, and if $x_0 = b$ we can choose $I = [b-\delta', b]$ where $\delta' = \min\{\delta/2, b-a\} > 0$.

(b) Prove that $\lim_{p\to\infty}\left(\int_a^b |f(x)|^p\,dx\right)^{1/p}$ exists and that

$$\lim_{p\to\infty}\left(\int_a^b |f(x)|^p\,dx\right)^{1/p} = M.$$

Proof. Let $\varepsilon > 0$. According to the definition of $\lim_{p\to\infty}\left(\int_a^b |f(x)|^p\,dx\right)^{1/p} = M$, we must show that we can find a $P \in \mathbb{R}$ so that

$$\left|\left(\int_a^b |f(x)|^p\; dx\right)^{1/p} - M\right| < \varepsilon \qquad \text{for } p > P. \tag{7}$$

From part (a) we can find a nondegenerate interval $I = I_{\varepsilon/2} \subset [a,b]$ so that

$$\left(M - \frac{\varepsilon}{2}\right)^p |I| \le \int_a^b |f(x)|^p\; dx \le M^p(b-a).$$

Raising each term above to the $1/p$ power and using that $x^{1/p}$ is an increasing function on $[0,+\infty)$, we can conclude that

$$\left(M - \frac{\varepsilon}{2}\right)|I|^{1/p} \le \left(\int_a^b |f(x)|^p\; dx\right)^{1/p} \le M(b-a)^{1/p}. \tag{8}$$

Since $b - a > 0$ we know that $\lim_{p\to\infty}(b-a)^{1/p} = 1$. Therefore, we can find a $P_1 \in \mathbb{R}$ so that

$$\left|(b-a)^{1/p} - 1\right| < \frac{\varepsilon}{M} \qquad \text{for } p > P_1$$

from which we can conclude

$$(b-a)^{1/p} < 1 + \frac{\varepsilon}{M} = \frac{M+\varepsilon}{M} \qquad \text{for } p > P_1. \tag{9}$$

Similarly, since $|I| > 0$, we know that $\lim_{p\to\infty}|I|^{1/p} = 1$ so consequently, there exists a $P_2 \in \mathbb{R}$ so that

$$\left||I|^{1/p} - 1\right| < \frac{\varepsilon/2}{M - \varepsilon/2} \qquad \text{for } p > P_2$$

from which we can conclude

$$|I|^{1/p} > 1 - \frac{\varepsilon/2}{M-\varepsilon/2} = \frac{M-\varepsilon}{M-\varepsilon/2} \qquad \text{for } p > P_2. \tag{10}$$

Defining $P = \max\{P_1, P_2\}$ and combining (8), (9), and (10) let's us conclude that

$$M - \varepsilon < \left(\int_a^b |f(x)|^p\; dx\right)^{1/p} < M + \varepsilon \qquad \text{for } p > P,$$

which is equivalent to (7).

$\square$