Math 421, Homework #2
Due: Wednesday, February 3

Homework assignments should be submitted during lecture. Please consult the homework rules and guidelines before completing the assignment. In particular:

- Write in compete sentences, organized into paragraphs.
- Leave plenty of room in between problems.
- Write only on the front side of each sheet of paper.
- Staple!
- Write on your assignment the names of any persons or sources consulted during its completion (other than the course text or instructor).

(1) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Assume that f has a finite number of discontinuities, i.e. assume there exists a finite subset E of $[a, b]$ so that f is continuous at all $x \in [a, b] \setminus E$. Prove that f is integrable on $[a, b]$.

(2) (a) Consider a function $f : [a, b] \to \mathbb{R}$, and assume that there is a single point at which f is nonzero, i.e. assume that there is a point $c \in [a, b]$ so that f satisfies $f(x) = 0$ for all $x \in [a, b] \setminus \{c\}$. Prove that f is integrable on $[a, b]$ and that $\int_a^b f(x)\,dx = 0$.

(b) Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Let $g : [a, b] \to \mathbb{R}$ be a function which agrees with f at all points in $[a, b]$ except for one, i.e. assume there exists a $c \in [a, b]$ so that $g(x) = f(x)$ for all $x \in [a, b] \setminus \{c\}$. Prove that g is integrable on $[a, b]$ and that $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$.

(c) (5.1.6) Let $f : [a, b] \to \mathbb{R}$ be an integrable function, and assume that $g : [a, b] \to \mathbb{R}$ agrees with f except on a finite set, i.e. assume there exists a finite set E so that $g(x) = f(x)$ for all $x \in [a, b] \setminus E$. Prove that g is integrable on $[a, b]$ and that $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$.

(3) (5.2.5) Prove that if f is integrable on $[0, 1]$ and $\beta > 0$, then

$$\lim_{n\to\infty} n^\alpha \int_0^{1/n^\beta} f(x)\,dx = 0$$

for all $\alpha < \beta$.

(4) (5.2.8) Let f be continuous on a closed, nondegenerate interval[1] $[a, b]$, let

$$M = \sup_{x\in[a,b]} |f(x)|,$$

and assume that $M > 0$.

(a) Prove that if $p > 0$, then for every $\varepsilon \in (0, M]$ there is a nondegenerate interval $I_\varepsilon \subset [a, b]$ such that

$$(M - \varepsilon)^p |I_\varepsilon| \le \int_a^b |f(x)|^p\,dx \le M^p(b - a)$$

where $|I_\varepsilon|$ denotes the length of the interval I_ε.

(b) Prove that $\lim_{p\to\infty} \left(\int_a^b |f(x)|^p\,dx\right)^{1/p}$ exists and that

$$\lim_{p\to\infty} \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} = M.$$

(Be careful: the I_ε from part (a) depends on ε, and $|I_\varepsilon|$ may approach 0 as $\varepsilon \to 0$.)

[1] That is to say $b > a$.