Math 421, Homework #1 Solutions

(1) Let $f,\, g : [a, b] \to \mathbb{R}$ be bounded functions.

(a) Show that

$$(U)\int_a^b (f(x)+g(x))\,dx \le (U)\int_a^b f(x)\,dx + (U)\int_a^b g(x)\,dx.$$

(This is part of problem 5.1.7(a) in the textbook.)

Proof. We start by proving a general fact about the supremum of a sum of functions that we will need. Let $A \subset [a, b]$. It is immediate from the definition of supremum that for any $x \in A$,

$$f(x) \le \sup_{x\in A} f(x) \qquad \text{and} \qquad g(x) \le \sup_{x\in A} g(x),$$

so adding these two inequalities together we get that

$$f(x)+g(x) \le \sup_{x\in A} f(x) + \sup_{x\in A} g(x).$$

for all $x \in A$. Therefore $\sup_{x\in A} f(x)+\sup_{x\in A} g(x)$ is an upper bound for the values of $f(x)+g(x)$ for $x \in A$. Again using the definition of supremum, we can conclude that

$$\sup_{x\in A}[f(x)+g(x)] \le \sup_{x\in A} f(x) + \sup_{x\in A} g(x). \tag{1}$$

Now, let $P = \{x_0, \dots, x_n\}$ of $[a, b]$. Using (1) we can conclude that

$$\begin{aligned} M_j(f+g) &= \sup_{x\in[x_{j-1},x_j]} (f(x)+g(x)) \\ &\le \sup_{x\in[x_{j-1},x_j]} f(x) + \sup_{x\in[x_{j-1},x_j]} g(x) \\ &= M_j(f)+M_j(g). \end{aligned}$$

Multiplying this inequality be $x_j - x_{j-1} (\ge 0)$ and summing from $j = 1$ to n allows us to conclude that

$$U(f+g,P) \le U(f,P) + U(g,P),$$

and since $(U)\int_a^b f(x)+g(x)\,dx \le U(f+g,P)$, we can conclude that

$$(U)\int_a^b f(x)+g(x)\,dx \le U(f,P)+U(g,P) \tag{2}$$

for any partition P of $[a, b]$.

Next, choose an $\varepsilon > 0$. Using the approximation property of infima, we can find a partition P_f of $[a, b]$ so that

$$U(f,P_f) < (U)\int_a^b f(x)\,dx + \varepsilon$$

and we can find a partition P_g of $[a, b]$ so that

$$U(g,P_g) < (U)\int_a^b g(x)\,dx + \varepsilon.$$

Defining $P = P_f \cup P_g$ to be a common refinement of P_f of P_g, we use Remark 5.7 to conclude that

$$U(f,P)+U(g,P) \le U(f,P_f)+U(g,P_g) < (U)\int_a^b f(x)\,dx + (U)\int_a^b g(x)\,dx + 2\varepsilon.$$

Combining this inequality with (2) we conclude that

$$(U)\int_a^b f(x)+g(x)\,dx < (U)\int_a^b f(x)\,dx + (U)\int_a^b g(x)\,dx + 2\varepsilon.$$

Since we can carry out this argument for any $\varepsilon > 0$, this implies that

$$(U)\int_a^b f(x) + g(x)\,dx \leq (U)\int_a^b f(x)\,dx + (U)\int_a^b g(x)\,dx$$

as claimed. $\square$

(b) Let $\alpha \geq 0$ be a constant. Show that

$$(U)\int_a^b \alpha f(x)\,dx = \alpha\left[(U)\int_a^b f(x)\,dx\right]$$

and that

$$(L)\int_a^b \alpha f(x)\,dx = \alpha\left[(L)\int_a^b f(x)\,dx\right].$$

Proof. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Using Theorem 7 part (4) from the sup/inf review sheet, we have that

$$\begin{aligned} M_j(\alpha f) &= \sup\{\alpha f(x) \mid x \in [x_{j-1}, x_j]\} \\ &= \sup \alpha\{f(x) \mid x \in [x_{j-1}, x_j]\} \\ &= \alpha \sup\{f(x) \mid x \in [x_{j-1}, x_j]\} \\ &= \alpha M_j(f). \end{aligned}$$

Multiplying by $(x_j - x_{j-1})$ and summing, we can conclude that

$$U(\alpha f, P) = \alpha U(f, P)$$

for any partition P of $[a, b]$, which is equivalent to saying that

$$\begin{aligned} \{U(\alpha f, P) \mid P \text{ a partition of } [a, b]\} &= \{\alpha U(f, P) \mid P \text{ a partition of } [a, b]\} \\ &= \alpha\{U(f, P) \mid P \text{ a partition of } [a, b]\}. \end{aligned}$$

Consequently, we can again use Theorem 7 part (4) to conclude that

$$\begin{aligned} (U)\int_a^b \alpha f(x)\,dx &= \inf\{U(\alpha f, P) \mid P \text{ a partition of } [a, b]\} \\ &= \inf(\alpha\{U(f, P) \mid P \text{ a partition of } [a, b]\}) \\ &= \alpha \inf\{U(f, P) \mid P \text{ a partition of } [a, b]\} \\ &= \alpha\left[(U)\int_a^b f(x)\,dx\right]. \end{aligned}$$

An analogous argument can be used to show that for any partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ that

$$m_j(\alpha f) = \alpha m_j(f)$$

and thus

$$L(\alpha f, P) = \alpha L(f, P).$$

Taking suprema on both sides as above and using Theorem 7 part (4) then let's us conclude that

$$(L)\int_a^b \alpha f(x)\,dx = \alpha\left[(L)\int_a^b f(x)\,dx\right].$$

$\square$

(c) Let $\alpha < 0$ be a constant. Show that

$$(U)\int_a^b \alpha f(x)\,dx = \alpha\left[(L)\int_a^b f(x)\,dx\right] \tag{3}$$

and that

$$(L)\int_a^b \alpha f(x)\,dx = \alpha\left[(U)\int_a^b f(x)\,dx\right]. \tag{4}$$

Proof. We first consider the case $\alpha = -1$. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$. Using Theorem 7 part (4) from the sup/inf review sheet, we have that

$$\begin{aligned} M_j(-f) &= \sup\{-f(x) \mid x \in [x_{j-1}, x_j]\} \\ &= \sup\left((-1)\{f(x) \mid x \in [x_{j-1}, x_j]\}\right) \\ &= -\inf\{f(x) \mid x \in [x_{j-1}, x_j]\} \\ &= -m_j(f). \end{aligned}$$

Multiplying by $(x_j - x_{j-1})$ and summing then let's us conclude that

$$U(-f, P) = -L(f, P).$$

Again applying Theorem 7 part (4), we have that

$$\begin{aligned} (U)\int_a^b -f(x)\,dx &= \inf\{U(-f,P) \mid P \text{ a partition of } [a,b]\} \\ &= \inf\{-L(f,P) \mid P \text{ a partition of } [a,b]\} \\ &= \inf\left((-1)\{L(f,P) \mid P \text{ a partition of } [a,b]\}\right) \\ &= -\sup\{L(f,P) \mid P \text{ a partition of } [a,b]\} \\ &= -(L)\int_a^b f(x)\,dx. \end{aligned}$$

For arbitrary $\alpha < 0$ we use the above with part (b) to write

$$\begin{aligned} (U)\int_a^b \alpha f(x)\,dx &= (U)\int_a^b -|\alpha| f(x)\,dx = -(L)\int_a^b |\alpha| f(x)\,dx \\ &= -|\alpha|\left[(L)\int_a^b f(x)\,dx\right] = \alpha\left[(L)\int_a^b f(x)\,dx\right], \end{aligned}$$

which proves (3). To prove (4), we apply (3) to $f_1 := \alpha f$ and $\alpha_1 := \frac{1}{\alpha} < 0$ to find

$$\begin{aligned} (U)\int_a^b f(x)\,dx &= (U)\int_a^b \tfrac{1}{\alpha}(\alpha f(x))\,dx = (U)\int_a^b \alpha_1 f_1(x)\,dx \\ &= \alpha_1\left[(L)\int_a^b f_1(x)\,dx\right] = \tfrac{1}{\alpha}\left[(L)\int_a^b \alpha f(x)\,dx\right] \end{aligned}$$

which yields (4) after multiplying through by α. $\square$

(d) Show that

$$(L)\int_a^b (f(x)+g(x))\,dx \geq (L)\int_a^b f(x)\,dx + (L)\int_a^b g(x)\,dx.$$

(This is the other part of problem 5.1.7(a) in the textbook. A short proof can be constructed using parts (a) and (c)).

Proof. Using part (a) on the bounded functions $-f$ and $-g$ we have that

$$(U)\int_a^b (-f(x)) + (-g(x))\,dx \leq (U)\int_a^b (-f(x))\,dx + (U)\int_a^b (-g(x))\,dx.$$

By part (c) we know that

$$(U)\int_a^b (-f(x)) + (-g(x))\,dx = -(L)\int_a^b f(x)+g(x)\,dx,$$

$$(U)\int_a^b (-f(x))\,dx = -(L)\int_a^b f(x)\,dx,$$

and

$$(U)\int_a^b (-g(x))\,dx = -(L)\int_a^b g(x)\,dx.$$

Substituting into the above inequality yields

$$-(L)\int_a^b (f(x)+g(x))\,dx \leq -(L)\int_a^b f(x)\,dx + -(L)\int_a^b g(x)\,dx.$$

which yields

$$(L)\int_a^b (f(x)+g(x))\,dx \geq (L)\int_a^b f(x)\,dx + (L)\int_a^b g(x)\,dx.$$

after multiplying both sides by -1. $\square$

(2) Let $f,\ g : [a,b] \to \mathbb{R}$ be integrable functions, and let $\alpha \in \mathbb{R}$ be a constant. Use problem (1) and Theorem 5.15 to prove:

(a) that $f+g$ is integrable on $[a,b]$ and that

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Proof. The assumption that f and g are integrable imply by Theorem 5.15 that

$$(U)\int_a^b f(x)\,dx = (L)\int_a^b f(x)\,dx = \int_a^b f(x)\,dx \tag{5}$$

and

$$(U)\int_a^b g(x)\,dx = (L)\int_a^b g(x)\,dx = \int_a^b g(x)\,dx.$$

Problem (1) parts (a) and (d) with Remark 5.14 then yield

$$\begin{aligned}\int_a^b f(x)\,dx + \int_a^b g(x)\,dx &= (L)\int_a^b f(x)\,dx + (L)\int_a^b g(x)\,dx \\ &\le (L)\int_a^b (f(x)+g(x))\,dx \\ &\le (U)\int_a^b (f(x)+g(x))\,dx \\ &\le (U)\int_a^b f(x)\,dx + (U)\int_a^b g(x)\,dx \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.\end{aligned}$$

We conclude that

$$(L)\int_a^b (f(x)+g(x))\,dx = (U)\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Theorem 5.15 then implies that $f+g$ is integrable on $[a,b]$ and that

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

$\square$

(b) that αf is integrable on $[a,b]$ and that

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$$

(consider the cases $\alpha \geq 0$ and $\alpha < 0$ separately).

Proof. Assume that $\alpha \geq 0$. Using Problem (1) part (b) with (5) lets us conclude

$$(U)\int_a^b \alpha f(x)\,dx = \alpha\left[(U)\int_a^b f(x)\,dx\right] = \alpha \int_a^b f(x)\,dx = \alpha\left[(L)\int_a^b f(x)\,dx\right] = (L)\int_a^b \alpha f(x)\,dx$$

so Theorem 5.15 let's us conclude that αf is integrable on $[a,b]$ and that

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx.$$

If $\alpha < 0$ we can similarly use Problem (1) part (c) with (5) to find

$$(U)\int_a^b \alpha f(x)\,dx = \alpha\left[(L)\int_a^b f(x)\,dx\right] = \alpha \int_a^b f(x)\,dx = \alpha\left[(U)\int_a^b f(x)\,dx\right] = (L)\int_a^b \alpha f(x)\,dx.$$

We again conclude that αf is integrable on $[a,b]$ and that

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx.$$

□

(3) Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases}$$

prove that f is integrable on $[0,1]$.

Proof. Let $\varepsilon > 0$. The function f is continuous on the interval $[\frac{\varepsilon}{4}, 1]$, and hence is integrable on $[\frac{\varepsilon}{4}, 1]$ (Theorem 5.10). We can therefore find a partition $P_\varepsilon = \{x_1 = \varepsilon/4, x_2, \ldots, x_n = 1\}$ of $[\varepsilon/4, 1]$ so that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon/2.$$

Define a partition P of $[0,1]$ by $P = P_\varepsilon \cup \{0\} = \{0, \varepsilon/4, x_2, \ldots, 1\}$. We then find that

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{j=1}^{n} (M_j(f) - m_j(f))(x_j - x_{j-1}) \\ &= (M_1(f) - m_1(f))(x_1 - x_0) + \sum_{j=2}^{n} (M_j(f) - m_j(f))(x_j - x_{j-1}) \\ &= \left(\sup_{x \in [0,\varepsilon/4]} f(x) - \inf_{x \in [0,\varepsilon/4]} f(x) \right) (\varepsilon/4 - 0) + U(f, P_\varepsilon) - L(f, P_\varepsilon). \end{aligned}$$

From the definition of f and the properties of the sin function, we know that $-1 \le f(x) \le 1$ for all $x \in [0,1]$. Moreover, choosing an $n \in \mathbb{N}$ satisfying $\frac{1}{n} < 2\pi\varepsilon/4$ we have that $f(\frac{1}{2\pi n + \frac{\pi}{2}}) = \sin(2\pi n + \frac{\pi}{2}) = 1$, and $f(\frac{1}{2\pi n + \frac{3\pi}{2}}) = \sin(2\pi n + \frac{3\pi}{2}) = -1$ so we conclude that

$$\sup_{x \in [0,\varepsilon/4]} f(x) = 1 \qquad \text{and} \qquad \inf_{x \in [0,\varepsilon/4]} f(x) = -1.$$

We thus conclude that

$$\begin{aligned} U(f,P) - L(f,P) &= \left(\sup_{x \in [0,\varepsilon/4]} f(x) - \inf_{x \in [0,\varepsilon/4]} f(x) \right) (\varepsilon/4 - 0) + U(f, P_\varepsilon) - L(f, P_\varepsilon) \\ &= 2\varepsilon/4 + U(f, P_\varepsilon) - L(f, P_\varepsilon) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus f is integrable on $[0,1]$. $\square$

(4) (5.1.4.a) Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and assume that there is a point $x_0 \in [a, b]$ so that f is continuous at x_0 and that $f(x_0) \neq 0$. Show that

$$(L) \int_a^b |f(x)|\, dx > 0.$$

(Be sure to clearly indicate how you use the assumption that f is continuous at x_0.)

Proof. We first observe that since the absolute value function is continuous on $\mathbb{R}$[1] that $|f|$ is continuous at x_0 since f is continuous at x_0. Therefore we can find a $\delta > 0$ so that if $x \in [a, b]$ and $|x - x_0| < \delta$ then

$$||f(x)| - |f(x_0)|| < \frac{1}{2}|f(x_0)|.$$

This is equivalent to

$$-\frac{1}{2}|f(x_0)| < |f(x)| - |f(x_0)| < \frac{1}{2}|f(x_0)|$$

from which we can conclude that

$$|f(x)| > \frac{1}{2}|f(x_0)| \qquad \text{for all } x \in [a, b] \cap (x_0 - \delta, x_0 + \delta). \tag{6}$$

To show that $(L) \int_a^b |f(x)|\, dx > 0$ it suffices to find a partition P so that $L(|f|, P) > 0$ since $(L) \int_a^b |f(x)|\, dx \geq L(|f|, P)$ for any partition P of $[a, b]$. We consider three cases:

Case 1: $x_0 \in (a, b)$. In this case, we choose a $\delta' < \delta$ small enough so that $[x_0 - \delta', x_0 + \delta'] \subset (a, b)$, and define a partition P of $[a, b]$ by $P = \{a, x_0 - \delta', x_0 + \delta', b\}$. Then, since $|f(x)| \geq 0$ for all $x \in [a, b]$ we know that

$$\inf_{x \in [a, x_0 - \delta']} |f(x)| \geq 0 \qquad \text{and} \qquad \inf_{x \in [x_0 + \delta', b]} |f(x)| \geq 0.$$

Meanwhile, since $[x_0 - \delta', x_0 + \delta'] \subset [a, b] \cap (x_0 - \delta, x_0 + \delta)$ we can conclude from (6) that

$$\inf_{x \in [x_0 - \delta', x_0 + \delta']} |f(x)| \geq \frac{1}{2}|f(x_0)|.$$

We therefore find that

$$\begin{aligned} L(|f|, P) &= \left[\inf_{[a, x_0 - \delta']} |f|\right](x_0 - \delta' - a) + \left[\inf_{[x_0 - \delta', x_0 + \delta']} |f|\right]([x_0 + \delta'] - [x_0 - \delta']) \\ &\qquad + \left[\inf_{x \in [x_0 + \delta', b]} |f(x)|\right](b - [x_0 + \delta']) \\ &\geq 0(x_0 - \delta' - a) + \frac{1}{2}|f(x_0)| 2\delta' + 0(b - x_0 - \delta') \\ &= \delta' |f(x_0)| > 0. \end{aligned}$$

Thus $L(|f|, P) > 0$ and consequently $(L) \int_a^b |f(x)|\, dx > 0$.

Case 2: $x_0 = a$. In this case we choose a $\delta' < \min\{\delta, b - a\}$ and consider the partition $P = \{a, a + \delta', b\}$. Again, the fact that $[a, a + \delta'] \subset [a, b] \cap (a - \delta, a + \delta)$ used with (6) implies that

$$\inf_{x \in [a, a + \delta']} |f(x)| \geq \frac{1}{2}|f(x_0)|,$$

and thus

$$\begin{aligned} L(|f|, P) &= \left[\inf_{[a, a + \delta']} |f|\right](a + \delta' - a) + \left[\inf_{[a + \delta', b]} |f|\right](b - [a + \delta']) \\ &\geq \frac{1}{2}|f(x_0)\delta' + 0(b - a - \delta') = \frac{1}{2}|f(x_0)|\delta' > 0, \end{aligned}$$

[1] Uniform continuity of $h(x) = |x|$ on $\mathbb{R}$ follows from the triangle inequality

$$||x| - |y|| \leq |x - y|.$$

Given $\varepsilon > 0$ choose $\delta = \varepsilon$. Then $|x - y| < \delta$ implies that $|h(x) - h(y)| = ||x| - |y|| \leq |x - y| < \delta = \varepsilon$.

and thus $(L)\int_a^b |f(x)|\,dx > 0$ in this case.

Case 3: $x_0 = b$. Choose a $\delta' < \min\{\delta, b-a\}$ and let $P = \{a, b-\delta', b\}$. A similar argument to that in case 2 shows that

$$L(|f|, P) \geq \frac{1}{2}|f(x_0)|\delta' > 0$$

and thus $(L)\int_a^b |f(x)|\,dx > 0$ in this case as well.

$\square$