Math 421, Homework #10 Solutions

(1) (11.1.4) Assume that $f : [a, b] \times [c, d] \to \mathbb{R}$ is continuous and that $g : [a, b] \to \mathbb{R}$ is integrable. Prove that

$$F(y) = \int_a^b g(x)f(x, y)\,dx$$

is uniformly continuous on $[c, d]$.

Proof. Let $\varepsilon > 0$.

Since g is assumed to be integrable on $[a, b]$, g is bounded on $[a, b]$. Let $M > 0$ be a number satisfying

$$|g(x)| \leq M \quad \text{for all } x \in [a, b]. \tag{1}$$

Since f is assumed to be continuous on $[a, b] \times [c, d]$ and $[a, b] \times [c, d]$ is compact, it follows from Theorem 9.25 that f is uniformly continuous on $[a, b] \times [c, d]$. We can therefore choose a $\delta > 0$ so that for (x_1, y_1), $(x_2, y_2) \in [a, b] \times [c, d]$ with $\|(x_1, y_1) - (x_2, y_2)\| < \delta$

$$|f(x_1, y_1) - f(x_2, y_2)| < \frac{\varepsilon}{2M(b-a)}. \tag{2}$$

Then if $|y_1 - y_2| < \delta$ it follows that $\|(x, y_1) - (x, y_2)\| = |y_1 - y_2| < \delta$ for all $x \in [a, b]$ so we find that

$$\begin{aligned}
|F(y_1) - F(y_2)| &= \left| \int_a^b g(x)f(x, y_1)\,dx - \int_a^b g(x)f(x, y_2)\,dx \right| \\
&= \left| \int_a^b g(x)\left[f(x, y_1) - f(x, y_2)\right]\,dx \right| && \text{linearity of the integral} \\
&\leq \int_a^b \left|g(x)\left[f(x, y_1) - f(x, y_2)\right]\right|\,dx && \text{Theorem 5.22} \\
&\leq \int_a^b M \frac{\varepsilon}{2M(b-a)}\,dx && \text{Thm. 5.21 with (1) and (2).} \\
&= \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

Therefore F is uniformly continuous on $[c, d]$. □

(2) (11.2.6) Prove that if $\alpha > \frac{1}{2}$, then

$$f(x,y) = \begin{cases} |xy|^\alpha \log(x^2+y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is differentiable at $(0,0)$.

Proof. We have that

$$f(x,0) = \begin{cases} |x(0)|^\alpha \log(x^2+(0)^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

so $f(x,0) = 0$ for all $x \in \mathbb{R}$ and hence $f_x(0,0) = 0$. Similarly $f(0,y)$ is identically 0 so $f_y(0,0) = 0$.

Let $\varepsilon > 0$. Since $\alpha > 1/2$ we can use L'Hopital's rule[1] to show that $\lim_{t\to 0^+} t^{\alpha-1/2}\log t = 0$. We can therefore choose $\delta_1 > 0$ so that

$$t^{\alpha-\frac{1}{2}}\,|\log t| = \left|t^{\alpha-\frac{1}{2}}\log t - 0\right| < \varepsilon \quad \text{for } 0 < t < \delta_1.$$

Define $\delta = \sqrt{\delta_1}$. Then when $0 < \sqrt{x^2+y^2} = \|(x,y)\| < \delta$, it follows that $0 < x^2+y^2 < \delta^2 = \delta_1$. We can therefore use that $|x| \leq \sqrt{x^2+y^2}$, $|y| \leq \sqrt{x^2+y^2}$, and that $\alpha > 1/2 > 0$, to find for $0 < \|(x,y)\| < \delta$ that

$$\begin{aligned}
\frac{|f(x,y) - f(0,0) - Df(0,0)(x,y)|}{\|(x,y)\|} &= \frac{\left||xy|^\alpha \log(x^2+y^2) - 0 - (0,0)\cdot(x,y)\right|}{\|(x,y)\|} \\
&= \frac{\left||xy|^\alpha \log(x^2+y^2)\right|}{\sqrt{x^2+y^2}} \\
&\leq \frac{(x^2+y^2)^\alpha \left|\log(x^2+y^2)\right|}{\sqrt{x^2+y^2}} \\
&= (x^2+y^2)^{\alpha-\frac{1}{2}}\left|\log(x^2+y^2)\right| \\
&< \varepsilon.
\end{aligned}$$

Therefore

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - Df(0,0)(x,y)}{\|(x,y)\|} = 0$$

so f is differentiable at $(0,0)$.

□

[1] Since $\log t \to -\infty$ and $t^{1/2-\alpha} \to +\infty$ as $t \to 0^+$ L'Hopital's rule says that

$$\lim_{t\to 0^+} t^{\alpha-1/2}\log t = \lim_{t\to 0^+} \frac{\log t}{t^{1/2-\alpha}} = \lim_{t\to 0^+} \frac{\frac{d}{dt}(\log t)}{\frac{d}{dt}(t^{1/2-\alpha})}$$

as long as the limit on the right exists. We have

$$\frac{\frac{d}{dt}(\log t)}{\frac{d}{dt}(t^{1/2-\alpha})} = \frac{t^{-1}}{(\frac{1}{2}-\alpha)t^{1/2-\alpha-1}} = (1/2-\alpha)^{-1}t^{\alpha-1/2} \to 0 \quad \text{as } t \to 0^+ \text{ since } \alpha - 1/2 > 0$$

so $\lim_{t\to 0^+} t^{\alpha-1/2}\log t = 0$.

(3) Determine whether or not the function

$$f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is differentiable at $(0,0)$ and prove that your answer is correct.

Proof. We claim that f is not differentiable at $(0,0)$. We have that

$$f(x,0) = \begin{cases} \frac{x^3+(0)^3}{x^2+(0)^2} & x \neq 0 \\ 0 & x = 0 \end{cases} = x$$

so $f_x(0,0) = 1$, and a similar computation shows that $f_y(0,0) = 1$. If f were differentiable at $(0,0)$ we would have that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - Df(0,0)(x,y)}{\|(x,y)\|} = 0. \tag{3}$$

However for $(x,y) \neq (0,0)$ we have that

$$\begin{aligned} \frac{f(x,y) - f(0,0) - Df(0,0)(x,y)}{\|(x,y)\|} &= \frac{\frac{x^3+y^3}{x^2+y^2} - 0 - (1,1)\cdot(x,y)}{(x^2+y^2)^{1/2}} \\ &= \frac{\frac{x^3+y^3}{x^2+y^2} - x - y}{(x^2+y^2)^{1/2}} \\ &= \frac{x^3+y^3-x^3-xy^2-yx^2-y^3}{(x^2+y^2)^{3/2}} \\ &= \frac{-xy(x+y)}{(x^2+y^2)^{3/2}}. \end{aligned}$$

But then considering the sequence $(x_n, y_n) = (1/n, 1/n) \to (0,0)$ we have that

$$\lim_{n\to\infty} \frac{f(1/n,1/n) - f(0,0) - Df(0,0)(1/n,1/n)}{\|(1/n,1/n)\|} = \lim_{n\to\infty} \frac{-(1/n)(1/n)(1/n+1/n)}{(1/n^2+1/n^2)^{3/2}} = -2^{-1/2}$$

so the sequential characterization of limits tells us that (3) can't be true. Therefore f is not differentiable at $(0,0)$. $\square$

(4) (11.2.8) Consider a linear transformation $\mathbf{T} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Prove that $\mathbf{T}$ is differentiable on $\mathbb{R}^n$ and that

$$D\mathbf{T}(\mathbf{a}) = \mathbf{T} \quad \text{ for all } \mathbf{a} \in \mathbb{R}^n.$$

Proof. Let $\mathbf{a} \in \mathbb{R}^n$. Using that $\mathbf{T}$ is linear, we have for any $\mathbf{h} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ that

$$\begin{aligned} \frac{\mathbf{T}(\mathbf{a}+\mathbf{h}) - \mathbf{T}(\mathbf{a}) - \mathbf{T}(\mathbf{h})}{\|\mathbf{h}\|} &= \frac{\mathbf{T}(\mathbf{a}) + \mathbf{T}(\mathbf{h}) - \mathbf{T}(\mathbf{a}) - \mathbf{T}(\mathbf{h})}{\|\mathbf{h}\|} \\ &= \frac{\mathbf{0}}{\|\mathbf{h}\|} = \mathbf{0}, \end{aligned}$$

and therefore

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{\mathbf{T}(\mathbf{a}+\mathbf{h}) - \mathbf{T}(\mathbf{a}) - \mathbf{T}(\mathbf{h})}{\|\mathbf{h}\|} = \mathbf{0}.$$

The definition of differentiable then let's us conclude that $\mathbf{T}$ is differentiable at $\mathbf{a}$ and that $D\mathbf{T}(\mathbf{a}) = \mathbf{T}$. □

(5) [The Quotient Rule] Let $V \subset \mathbb{R}^n$ be an open set and assume that $f,\ g : V \to \mathbb{R}$ are differentiable at $\mathbf{a} \in V$ and that $g(\mathbf{a}) \neq 0$. Prove that f/g is defined on an open ball containing $\mathbf{a}$, that f/g is differentiable at $\mathbf{a}$ and that

$$D\left(\frac{f}{g}\right)(\mathbf{a}) = \frac{1}{[g(\mathbf{a})]^2}\left(g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})\right)$$

(see problem 11.3.6 for some guidance about how to break this down into smaller steps).

Proof. First we note that since g is differentiable at $\mathbf{a}$, g is continuous at $\mathbf{a}$. Since we assume that $g(\mathbf{a}) \neq 0$ it follows from problem (2) in homework #9 that there exists an $\varepsilon > 0$ so that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in B_\varepsilon(\mathbf{a})$. Therefore $\frac{1}{g(\mathbf{x})}$ and hence $\frac{f(\mathbf{x})}{g(\mathbf{x})}$ is defined for all $\mathbf{x} \in B_\varepsilon(\mathbf{a})$.

We next claim that $\frac{1}{g}$ is differentiable at $\mathbf{a}$ and that $D\left(\frac{1}{g}\right)(\mathbf{a}) = -\frac{1}{g(\mathbf{a})^2}Dg(\mathbf{a})$. Indeed since g is assumed to be differentiable at $\mathbf{a}$ we have that

$$R(\mathbf{x}) := \frac{g(\mathbf{x}) - g(\mathbf{a}) - Dg(\mathbf{a})(\mathbf{x}-\mathbf{a})}{\|\mathbf{x}-\mathbf{a}\|} \to 0 \quad \text{as } \mathbf{x} \to \mathbf{a}.$$

We therefore find that for $\mathbf{x} \neq \mathbf{a}$ that

$$\begin{aligned}
\frac{\frac{1}{g(\mathbf{x})} - \frac{1}{g(\mathbf{a})} - \left[-\frac{1}{g(\mathbf{a})^2}Dg(\mathbf{a})\right](\mathbf{x}-\mathbf{a})}{\|\mathbf{x}-\mathbf{a}\|} &= \frac{\frac{1}{g(\mathbf{x})} - \frac{1}{g(\mathbf{a})} - \left[-\frac{1}{g(\mathbf{a})^2}Dg(\mathbf{a})\right](\mathbf{x}-\mathbf{a})}{\|\mathbf{x}-\mathbf{a}\|} \\
&= \frac{g(\mathbf{a})^2 - g(\mathbf{x})g(\mathbf{a}) + g(\mathbf{x})Dg(\mathbf{a})(\mathbf{x}-\mathbf{a})}{g(\mathbf{a})^2 g(\mathbf{x})\,\|\mathbf{x}-\mathbf{a}\|} \\
&= \frac{[g(\mathbf{x}) - g(\mathbf{a})]\,Dg(\mathbf{a})(\mathbf{x}-\mathbf{a}) - g(\mathbf{a})\,[g(\mathbf{x}) - g(\mathbf{a}) - Dg(\mathbf{a})(\mathbf{x}-\mathbf{a})]}{g(\mathbf{a})^2 g(\mathbf{x})\,\|\mathbf{x}-\mathbf{a}\|} \\
&= \tfrac{g(\mathbf{x})-g(\mathbf{a})}{g(\mathbf{a})^2 g(\mathbf{x})}Dg(\mathbf{a})\tfrac{\mathbf{x}-\mathbf{a}}{\|\mathbf{x}-\mathbf{a}\|} - \left(\tfrac{1}{g(\mathbf{a})g(\mathbf{x})}\right)\tfrac{g(\mathbf{x})-g(\mathbf{a})-Dg(\mathbf{a})(\mathbf{x}-\mathbf{a})}{\|\mathbf{x}-\mathbf{a}\|} \\
&=: T_1(\mathbf{x}) - \tfrac{1}{g(\mathbf{a})g(\mathbf{x})}R(\mathbf{x}).
\end{aligned}$$

We would like to show that this quantity approaches 0 as $\mathbf{x} \to \mathbf{a}$. We know that $R(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{a}$, and hence that $\frac{1}{g(\mathbf{a})g(\mathbf{x})}R(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{a}$ since g is continuous at $\mathbf{a}$ and therefore $g(\mathbf{x}) \to g(\mathbf{a}) \neq 0$ as $\mathbf{x} \to \mathbf{a}$. To understand the first term, $T_1(\mathbf{x})$, we use the properties of the operator norm[2] to conclude that

$$\begin{aligned}
|T_1(\mathbf{x})| &= \left|\frac{g(\mathbf{x}) - g(\mathbf{a})}{g(\mathbf{a})^2 g(\mathbf{x})}Dg(\mathbf{a})\frac{\mathbf{x}-\mathbf{a}}{\|\mathbf{x}-\mathbf{a}\|}\right| \\
&\leq \left|\frac{g(\mathbf{x}) - g(\mathbf{a})}{g(\mathbf{a})^2 g(\mathbf{x})}\right| \|Dg(\mathbf{a})\| \left\|\frac{\mathbf{x}-\mathbf{a}}{\|\mathbf{x}-\mathbf{a}\|}\right\| \\
&= \left|\frac{g(\mathbf{x}) - g(\mathbf{a})}{g(\mathbf{a})^2 g(\mathbf{x})}\right| \|Dg(\mathbf{a})\|.
\end{aligned}$$

Again using the continuity of g at $\mathbf{a}$ and that $g(\mathbf{a}) \neq 0$, we can use the above inequality with the squeeze theorem to conclude that $T_1(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{a}$. Therefore

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{1/g(\mathbf{x}) - 1/g(\mathbf{a}) - \left[-\frac{1}{g(\mathbf{a})^2}Dg(\mathbf{a})\right](\mathbf{x}-\mathbf{a})}{\|\mathbf{x}-\mathbf{a}\|} = 0$$

so we can conclude that $1/g$ is differentiable at $\mathbf{a}$ and that $D(1/g)(\mathbf{a}) = -(1/g(\mathbf{a})^2)Dg(\mathbf{a})$.

[2] Since g is real-valued we could also use that $Dg(\mathbf{a}) = \nabla g(\mathbf{a})^T$ and then use the Cauchy-Schwartz inequality. For any $\mathbf{v} \in \mathbb{R}^n$ we would have that

$$|Dg(\mathbf{a})\mathbf{v}| = \left|\nabla g(\mathbf{a})^T\mathbf{v}\right| = |\nabla g(\mathbf{a}) \cdot \mathbf{v}| \leq \|\nabla g(\mathbf{a})\|\,\|\mathbf{v}\|.$$

Finally, we apply the special case of Theorem 11.20 (equation (9)) when the codomain is 1-dimensional to find that $f/g = f(1/g)$ is differentiable at $\mathbf{a}$ and that

$$\begin{aligned} D\left(f/g\right)(\mathbf{a}) &= D\left((f)\cdot\left(\frac{1}{g}\right)\right)(\mathbf{a}) \\ &= \frac{1}{g(\mathbf{a})}Df(\mathbf{a}) + f(\mathbf{a})D\left(\frac{1}{g}\right)(\mathbf{a}) \\ &= \frac{1}{g(\mathbf{a})}Df(\mathbf{a}) - f(\mathbf{a})\frac{1}{g(\mathbf{a})^2}Dg(\mathbf{a}) \\ &= \frac{1}{g(\mathbf{a})^2}\left[g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})\right]. \end{aligned}$$

□