Math 421, Homework #10
Due: Wednesday, April 21

Homework assignments should be submitted during lecture. Please consult the homework rules and guidelines before completing the assignment. In particular:

- Write in compete sentences, organized into paragraphs.
- Leave plenty of room in between problems.
- Write only on the front side of each sheet of paper.
- Staple!
- Write on your assignment the names of any persons or sources consulted during its completion (other than the course text or instructor).

(1) (11.1.4) Assume that $f : [a, b] \times [c, d] \to \mathbb{R}$ is continuous and that $g : [a, b] \to \mathbb{R}$ is integrable. Prove that

$$F(y) = \int_a^b g(x) f(x, y)\, dx$$

is uniformly continuous on $[c, d]$.

(2) (11.2.6) Prove that if $\alpha > \frac{1}{2}$, then

$$f(x, y) = \begin{cases} |xy|^\alpha \log(x^2 + y^2) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is differentiable at $(0, 0)$.

(3) Determine whether or not the function

$$f(x, y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is differentiable at $(0, 0)$ and prove that your answer is correct.

(4) (11.2.8) Consider a linear transformation $\mathbf{T} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Prove that $\mathbf{T}$ is differentiable on $\mathbb{R}^n$ and that

$$D\mathbf{T}(\mathbf{a}) = \mathbf{T} \quad \text{for all } \mathbf{a} \in \mathbb{R}^n.$$

(5) [The Quotient Rule] Let $V \subset \mathbb{R}^n$ be an open set and assume that $f,\ g : V \to \mathbb{R}$ are differentiable at $\mathbf{a} \in V$ and that $g(\mathbf{a}) \neq 0$. Prove that f/g is defined on an open ball containing $\mathbf{a}$, that f/g is differentiable at $\mathbf{a}$ and that

$$D\left(\frac{f}{g}\right)(\mathbf{a}) = \frac{1}{[g(\mathbf{a})]^2}\left(g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})\right)$$

(see problem 11.3.6 for some guidance about how to break this down into smaller steps).