Math 421, Homework #1
Due: Wednesday, January 27

Homework assignments should be submitted during lecture. Please consult the homework rules and guidelines before completing the assignment. In particular:

- Write in compete sentences, organized into paragraphs.
- Leave plenty of room in between problems.
- Write only on the front side of each sheet of paper.
- Staple!
- Write on your assignment the names of any persons or sources consulted during its completion (other than the course text or instructor).

(1) Let $f,\, g : [a,b] \to \mathbb{R}$ be bounded functions.

(a) Show that
$$(U)\int_a^b (f(x)+g(x))\,dx \le (U)\int_a^b f(x)\,dx + (U)\int_a^b g(x)\,dx.$$
(This is part of problem 5.1.7(a) in the textbook.)

(b) Let $\alpha \ge 0$ be a constant. Show that
$$(U)\int_a^b \alpha f(x)\,dx = \alpha\left[(U)\int_a^b f(x)\,dx\right]$$
and that
$$(L)\int_a^b \alpha f(x)\,dx = \alpha\left[(L)\int_a^b f(x)\,dx\right].$$

(c) Let $\alpha < 0$ be a constant. Show that
$$(U)\int_a^b \alpha f(x)\,dx = \alpha\left[(L)\int_a^b f(x)\,dx\right]$$
and that
$$(L)\int_a^b \alpha f(x)\,dx = \alpha\left[(U)\int_a^b f(x)\,dx\right].$$

(d) Show that
$$(L)\int_a^b (f(x)+g(x))\,dx \ge (L)\int_a^b f(x)\,dx + (L)\int_a^b g(x)\,dx.$$
(This is the other part of problem 5.1.7(a) in the textbook. A short proof can be constructed using parts (a) and (c)).

(2) Let $f,\, g : [a,b] \to \mathbb{R}$ be integrable functions, and let $\alpha \in \mathbb{R}$ be a constant. Use problem (1) and Theorem 5.15 to prove:

(a) that $f+g$ is integrable on $[a,b]$ and that
$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

(b) that αf is integrable on $[a,b]$ and that
$$\int_a^b \alpha f(x)\,dx = \alpha\int_a^b f(x)\,dx$$
(consider the cases $\alpha \ge 0$ and $\alpha < 0$ separately).

(3) Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases}$$

prove that f is integrable on $[0,1]$.

(4) (5.1.4.a) Let $f : [a,b] \to \mathbb{R}$ be a bounded function, and assume that there is a point $x_0 \in [a,b]$ so that f is continuous at x_0 and that $f(x_0) \neq 0$. Show that

$$(L) \int_a^b |f(x)|\, dx > 0.$$

(Be sure to clearly indicate how you use the assumption that f is continuous at x_0.)