1. Determine whether each of the following statements is true or false. If a statement is false, provide a counterexample. (Write out the word "true" or "false" completely! No proof or explanation is necessary if you answer "true," and you don't need to prove your proposed counterexample is a counterexample if you answer "false.")

   (a) Let $f : E \subset \mathbf{R}^n \to \mathbf{R}^m$ be a continuous function. If $H \subset \mathbf{R}^m$ is compact, then $f^{-1}(H)$ is compact.

   Answer. False. Let $f : \mathbf{R} \to \mathbf{R}$, be defined by $f(x) = 0$ for all $x \in \mathbf{R}$. Then $\{0\} \in \mathbf{R}$ is a compact set, but $f^{-1}(\{0\}) = \mathbf{R}$ which is not compact. □

   (b) Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a continuous function. If $U \subset \mathbf{R}^n$ is an open set, then $f(U) \subset \mathbf{R}^m$ is an open set.

   Answer. False. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$. Then $(-1, 1)$ is open in $\mathbf{R}$, but $f((-1, 1)) = [0, 1)$ is not an open set. □

   (c) Let $E \subset \mathbf{R}^n$ be a closed set. Let $\{\mathbf{x}_k\}_{k\in\mathbf{N}}$ be a sequence with $\mathbf{x}_k \in E$ for all $k \in \mathbf{N}$, and assume $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}$. Then $\mathbf{x} \in E$.

   Answer. True. Since E is closed, E contains all its limit points (see Theorem 9.8), so $\mathbf{x} \in E$. □

   (d) Let U, $V \subset \mathbf{R}^n$ be connected sets. Then $U \cap V$ is connected.

   Answer. False. Let $f : \mathbf{R} \to \mathbf{R}^2$ be defined by $f(t) = (\cos t, \sin t)$, and let $U = f([0, \frac{3\pi}{2}])$, and $V = f([\pi, \frac{5\pi}{2}])$. Then U and V are connected, since they are each the continous image of a connected set (Theorem 9.30), but $U \cap V = f([0, \frac{\pi}{2}]) \cup f([\pi, \frac{3\pi}{2}])$ is not connected. □

2. (a) Define what it means for a set $E \subset \mathbf{R}^n$ to be connected.

Answer. A set $E \subset \mathbf{R}^n$ is said to be disconnected if there exist subsets $U \subset E$, and $V \subset E$ so that:

- U and V are each nonempty,
- U and V are each relatively open in E,
- $U \cup V = E$, and
- $U \cap V = \emptyset$.

If E is not disconnected, then E is said to be connected. □

(b) Assume that $E \subset \mathbf{R}^n$ is connected. Prove that the closure $\overline{E}$ is also connected.

Proof. Assume that $\overline{E}$ is not connected. Then there exist subsets U, V of $\overline{E}$ so that U and V are disjoint ($U \cap V = \emptyset$), nonempty, relatively open in $\overline{E}$, and so that $\overline{E} = U \cup V$.

Define $U' = E \cap U$, and $V' = E \cap V$. We claim that U' and V' are nonempty, relatively open in E, and satisfy $E = U' \cup V'$, and $U' \cap V' = \emptyset$, and thus E is not connected if $\overline{E}$ is not connected. Indeed, using that $E \subset \overline{E}$, we find

$$U' \cup V' = (E \cap U) \cup (E \cap V) = E \cap (U \cup V) = E \cap \overline{E} = E$$

and further

$$U' \cap V' = (E \cap U) \cap (E \cap V) = E \cap (U \cap V) = E \cap \emptyset = \emptyset,$$

so $U' \cup V' = E$ and $U' \cap V' = \emptyset$, as claimed.

To see that U' is relatively open in E, we note that since U is relatively open in $\overline{E}$, there exists an open set $A \subset \mathbf{R}^n$ so that $U = \overline{E} \cap A$. Then,

$$U' = E \cap U = E \cap (\overline{E} \cap A) = (E \cap \overline{E}) \cap A = E \cap A$$

so U' is relatively open in E since A is open. An identical argument shows that V' is relatively open in E.

Finally we claim that U' is nonempty. We saw above that there is an open set A so that $U' = E \cap A$ and $U = \overline{E} \cap A \neq \emptyset$. Suppose that $U' = E \cap A$ is empty. Then $E \subset A^c$, and since A is open A^c is closed. But since A^c is a closed set containing E, Theorem 8.32 (iii) tells us that $\overline{E} \subset A^c$. This in turn implies that $U = \overline{E} \cap A = \emptyset$, in contradiction to the fact that $U \neq \emptyset$. Therefore U' is nonempty. An identical arguement shows that $V' = E \cap V$ is nonempty since V is nonempty and relatively open in $\overline{E}$.

In conclusion, we have shown that if $\overline{E}$ is not connected, then E is not connected. Therefore, if E is connected, $\overline{E}$ must be connected as well.

□

3. Let $E \subset \mathbf{R}^n$ be a closed set, and assume that $\mathbf{a} \notin E$. Prove that

$$\inf_{\mathbf{x} \in E} \|\mathbf{x} - \mathbf{a}\| > 0.$$

Proof. The assumption $\mathbf{a} \notin E$ is equivalent to $\mathbf{a} \in E^c$. Since E is closed, E^c is open, so there exists an $\varepsilon > 0$ so that $B_\varepsilon(\mathbf{a}) \subset E^c$. Therefore, given $\mathbf{x} \in E$, $\mathbf{x} \notin E^c$, and consequently $\mathbf{x} \notin B_\varepsilon(\mathbf{a})$. This implies that

$$\|\mathbf{x} - \mathbf{a}\| \geq \varepsilon \text{ for all } \mathbf{x} \in E$$

and we can conclude that

$$\inf_{\mathbf{x} \in E} \|\mathbf{x} - \mathbf{a}\| \geq \varepsilon > 0.$$

□

Alternative proof. If $E = \emptyset$, then

$$\inf_{\mathbf{x} \in E} \|\mathbf{x} - \mathbf{a}\| = +\infty > 0,$$

so the result holds in this case.

Assume that E is nonempty, and let $\mathbf{x}_k \in E$ be sequence so that

$$\lim_{k \to \infty} \|\mathbf{x}_k - \mathbf{a}\| = \inf_{\mathbf{x} \in E} \|\mathbf{x} - \mathbf{a}\|.$$

Then the sequence $\{\mathbf{x}_k\}$ is bounded, since

$$\|\mathbf{x}_k\| \leq \|\mathbf{x}_k - \mathbf{a}\| + \|\mathbf{a}\|$$

so

$$\sup_{k \in \infty} \|\mathbf{x}_k\| \leq \sup_{k \in \infty} \|\mathbf{x}_k - \mathbf{a}\| + \|\mathbf{a}\|$$

and the right hand side of this inequality is finite since

$$\lim_{k \to \infty} \|\mathbf{x}_k - \mathbf{a}\| = \inf_{\mathbf{x} \in E} \|\mathbf{x} - \mathbf{a}\|$$

which is finite since E is nonempty.

Applying the Bolzano-Weierstrass theorem, we can find a subsequence $\{\mathbf{x}_{k_j}\}$ and an $\mathbf{x} \in \mathbf{R}^n$ so that

$$\lim_{j \to \infty} \mathbf{x}_{k_j} = \mathbf{x}.$$

Further, we know that $\mathbf{x} \in E$ since E is closed, and therefore contains all of its limit points (Theorem 9.8). We then have that

$$\inf_{\mathbf{x} \in E} \|\mathbf{x} - \mathbf{a}\| = \lim_{k \to \infty} \|\mathbf{x}_k - \mathbf{a}\| = \lim_{j \to \infty} \|\mathbf{x}_{k_j} - \mathbf{a}\| = \| \lim_{j \to \infty} \mathbf{x}_{k_j} - \mathbf{a}\| = \|\mathbf{x} - \mathbf{a}\|$$

and we must have that $\|\mathbf{x} - \mathbf{a}\| > 0$ since $\mathbf{x} \in E$ and $\mathbf{a} \notin E$, so $\mathbf{x} \neq \mathbf{a}$. We therefore have that $\inf_{\mathbf{x} \in E} \|\mathbf{x} - \mathbf{a}\| > 0$ as claimed. □

4. Let $U \subset \mathbf{R}^n$ be an open set, and let $\mathbf{a} \in U$. Assume that the function $f : U \setminus \{\mathbf{a}\} \to \mathbf{R}^m$ is uniformly continuous on $U \setminus \{\mathbf{a}\}$. Prove that

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x})$$

exists. (Hint: Use the sequential characterization of limits.)

Proof. By the sequential characterization of limits, the limit

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x})$$

exists if there exists an $\mathbf{L} \in \mathbf{R}^m$ so that for every sequence $\{\mathbf{x}_k\}_{k \in \mathbf{N}}$ with $\mathbf{x}_k \in U \setminus \{\mathbf{a}\}$ and $\lim_{k \to \infty} \mathbf{x}_k = \mathbf{a}$, we have that

$$\lim_{k \to \infty} f(\mathbf{x}_k) = \mathbf{L}.$$

(Note that the fact that there exist sequences $\mathbf{x}_k \in U \setminus \{\mathbf{a}\}$ with $\lim_{k \to \infty} \mathbf{x}_k = \mathbf{a}$ follows from the fact that U is an open set.)

Choose a sequence $\mathbf{x}_k \in U \setminus \{\mathbf{a}\}$ satisfying $\lim_{k \to \infty} \mathbf{x}_k = \mathbf{a}$. We claim that $f(\mathbf{x}_k)$ is a Cauchy sequence, and therefore is convergent. To see this, choose $\varepsilon > 0$. Then, since f is uniformly continuous on $U \setminus \{\mathbf{a}\}$, we can find a $\delta > 0$ so that

$$\|\mathbf{x} - \mathbf{y}\| < \delta \text{ and } \mathbf{x}, \mathbf{y} \in U \setminus \{\mathbf{a}\} \Longrightarrow \|f(\mathbf{x}) - f(\mathbf{y})\| < \varepsilon. \tag{1}$$

Choosing such a $\delta > 0$, we then choose an N so that $k > N$ implies that

$$\|\mathbf{x}_k - \mathbf{a}\| < \delta/2.$$

Then for k, $j > N$, we have that

$$\|\mathbf{x}_k - \mathbf{x}_j\| = \|\mathbf{x}_k - \mathbf{a} + \mathbf{a} - \mathbf{x}_j\| \leq \|\mathbf{x}_k - \mathbf{a}\| + \|\mathbf{a} - \mathbf{x}_j\| < \delta/2 + \delta/2 = \delta$$

and therefore,

$$\|f(\mathbf{x}_k) - f(\mathbf{x}_j)\| < \varepsilon.$$

Therefore $f(\mathbf{x}_k)$ is a Cauchy sequence, and converges to some $\mathbf{L} \in \mathbf{R}^m$.

In order to complete the proof, we need to show that any other sequence $\mathbf{y}_k \in U \setminus \{\mathbf{a}\}$ with $\lim_{k \to \infty} \mathbf{y}_k = \mathbf{a}$, also satisfies $\lim_{k \to \infty} f(\mathbf{y}_k) = \mathbf{L}$. Let $\mathbf{y}_k \in U \setminus \{\mathbf{a}\}$ be a second sequence satisfying $\lim_{k \to \infty} \mathbf{y}_k = \mathbf{a}$. By the argument of the previous paragraph, the sequence $f(\mathbf{y}_k)$ converges. Defining

$$\mathbf{L}' = \lim_{k \to \infty} f(\mathbf{y}_k),$$

we need to show that $\mathbf{L} = \mathbf{L}'$, which, since

$$\mathbf{L} - \mathbf{L}' = \lim_{k \to \infty} f(\mathbf{x}_k) - \lim_{k \to \infty} f(\mathbf{y}_k) = \lim_{k \to \infty} f(\mathbf{x}_k) - f(\mathbf{y}_k)$$

is equivalent to showing $\lim_{k\to\infty} f(\mathbf{x}_k) - f(\mathbf{y}_k) = \mathbf{0}$. Given $\varepsilon > 0$, choose δ so that (1) holds, and choose N so that $k > N$ implies that

$$\|\mathbf{x}_k - \mathbf{a}\| < \delta/2 \text{ and } \|\mathbf{y}_k - \mathbf{a}\| < \delta/2.$$

Then, for $k > N$, we have that

$$\|\mathbf{x}_k - \mathbf{y}_k\| \leq \|\mathbf{x}_k - \mathbf{a}\| + \|\mathbf{a} - \mathbf{y}_k\| < \delta/2 + \delta/2 = \delta$$

which, by (1), implies that

$$\|f(\mathbf{x}_k) - f(\mathbf{y}_k)\| < \varepsilon.$$

We thus conclude that

$$\lim_{k\to\infty} f(\mathbf{x}_k) - f(\mathbf{y}_k) = \mathbf{0}$$

and hence $\mathbf{L} = \mathbf{L}'$ as claimed. $\square$