1. Determine whether each of the following statements is true or false. If a given statement is true, write the word "TRUE" (no explanation or proof is necessary). If a given statement is false, write the word "FALSE" and *provide a concrete counterexample* (you do not need to prove that your proposed counterexample is a counterexample).

   (a) Let E be a subset of $\mathbf{R}^n$ and assume that the interior E^o is connected. Then E is also connected.

   False. Let $E = (0,1) \cup \{2\}$. Then $E^o = (0,1)$ which is connected, but E is not connected. $\square$

   (b) Let $\{\mathbf{x}_n\}$ be a sequence in $\mathbf{R}^n$, and assume that $\{\mathbf{x}_n\}$ has a convergent subsequence. Then $\{\mathbf{x}_n\}$ is bounded.

   False. Let $\{x_k\}$ be the sequence in $\mathbf{R}$ defined by

   $$x_k = \begin{cases} k & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even.} \end{cases}$$

   Then the sequence $\{x_k\}$ is not bounded, but the subsequence $\{x_{2k}\}_{k \in \mathbf{N}}$ is convergent. $\square$

   (c) Let $\{\mathbf{x}_n\}$ be a bounded sequence in $\mathbf{R}^3$. Then $\{\mathbf{x}_n\}$ is convergent.

   False. Let $\mathbf{x}_n = ((-1)^n, 0, 0)$. Then $\|\mathbf{x}_n\| = 1$ for all n so the sequence is bounded, but is not convergent. $\square$

   (d) A closed subset of a compact set is compact.

   True. According to the Heine-Borel theorem, a set $K \subset \mathbf{R}^n$ is compact if and only if it is closed and bounded. Boundedness of K implies that there is an $R > 0$ so that $K \subset B_R(\mathbf{0})$, so if $E \subset K$, then $E \subset B_R(\mathbf{0})$, and hence E is bounded as well. Therefore if $E \subset K$ is closed, it will be closed and bounded, and therefore compact. $\square$

2. (a) Let $E \subset \mathbf{R}^n$. Give the definition of the *interior* E^o of E.

Definition. The interior E^o of E is the union of all open sets contained in E, i.e.

$$E^o = \bigcup_{\substack{V \subset E \\ V \text{ is open}}} V.$$

$\square$

(b) Let $E \subset \mathbf{R}^2$ be defined by

$$E = \left\{(x, y) \,|\, x^2 + y^2 < 1\right\} \cup \{(x, 0) \,|\, -2 \leq x < 2\} \cup \left\{(0, 3 + \tfrac{1}{n}) \,|\, n \in \mathbf{N}\right\},$$

and let

$$\mathbf{a} = (-\tfrac{3}{2}, 0) \qquad \mathbf{b} = (0, 0) \qquad \mathbf{c} = (2, 0) \qquad \mathbf{d} = (0, 4) \qquad \mathbf{e} = (0, 3)$$

Answer each of following (no explanation or proof).

i. Which of the points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, and $\mathbf{e}$ are elements of the interior E^o of E?

Answer. $E^o \cap \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}\} = \{\mathbf{b}\}$ $\square$

ii. Which of the points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, and $\mathbf{e}$ are elements of the closure $\bar{E}$ of E?

Answer. $\bar{E} \cap \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}\} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}\}$ $\square$

iii. Which of the points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, and $\mathbf{e}$ are elements of the boundary ∂E of E?

Answer. $\partial E \cap \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}\} = \{\mathbf{a}, \mathbf{c}, \mathbf{d}, \mathbf{e}\}$ $\square$

3. Recall that a *cluster point* of a set $E \subset \mathbf{R}^n$ is a point $\mathbf{a} \in \mathbf{R}^n$ satisfying

$$E \cap B_\varepsilon(\mathbf{a}) \setminus \{\mathbf{a}\} \neq \emptyset \quad \text{for all } \varepsilon > 0.$$

Assume that $E \subset \mathbf{R}^n$ is connected and that $\mathbf{a} \in \mathbf{R}^n$ is a cluster point of E. Show that $E \cup \{\mathbf{a}\}$ is connected.

Proof. We will argue by contradiction. Assume that $E \cap \{\mathbf{a}\}$ is not connected. Then there exist sets $U \subset E \cup \{\mathbf{a}\}$ and $V \subset E \cup \{\mathbf{a}\}$ separating $E \cup \{\mathbf{a}\}$, i.e. U and V are nonempty, U and V are relatively open in $E \cup \{\mathbf{a}\}$, $U \cap V = \emptyset$, and $U \cup V = E \cup \{\mathbf{a}\}$.

Define $U' = U \cap E$ and $V' = V \cap E$. We claim that U' and V' separate E. To see this, we first observe that since $U \cap V = \emptyset$ we have that

$$U' \cap V' = (E \cap U) \cap (E \cap V) = E \cap (U \cap V) = E \cap \emptyset = \emptyset.$$

Similarly, since $U \cup V = E \cup \{\mathbf{a}\}$, we have that

$$U' \cup V' = (U \cap E) \cup (V \cap E) = (U \cup V) \cap E = (E \cup \{\mathbf{a}\}) \cap E = E.$$

Thus $U' \cup V' = E$ and $U' \cap V' = \emptyset$.

We next claim that U' and V' are relatively open in E. Indeed, since U is relatively open in $E \cup \{\mathbf{a}\}$, we have that $U = A \cap (E \cup \{\mathbf{a}\})$ for some open set A. Then,

$$U' = U \cap E = A \cap (E \cup \{\mathbf{a}\}) \cap E = A \cap E$$

so U' is relatively open in E since it can be written an open set intersected with E. An identical argument shows that V' is relatively open in E as well.

Finally, we claim that U' and V' are nonempty. Arguing by contradiction, assume that U' is empty. Then since $U \subset E \cup \{\mathbf{a}\}$ is nonempty, and $U' = U \cap E = \emptyset$, we must have that $U = \{\mathbf{a}\}$. But, since U is assumed to be relatively open in $E \cup \{\mathbf{a}\}$, Remark 8.27 tells us that there exists an $\varepsilon > 0$ so that

$$B_\varepsilon(\mathbf{a}) \cap (E \cap \{\mathbf{a}\}) \subset U = \{\mathbf{a}\}.$$

This in turn implies that

$$\begin{aligned} B_\varepsilon(\mathbf{a}) \cap E \setminus \{\mathbf{a}\} &= B_\varepsilon(\mathbf{a}) \cap (E \cup \{\mathbf{a}\}) \setminus \{\mathbf{a}\} \\ &\subset \{\mathbf{a}\} \setminus \{\mathbf{a}\} = \emptyset, \end{aligned}$$

and hence

$$E \cap B_\varepsilon(\mathbf{a}) \setminus \{\mathbf{a}\} = \emptyset.$$

This contradicts the assumption that $\mathbf{a}$ is a cluster point of E, and this contradiction shows that U' is nonempty. An identical argument shows that V' is nonempty.

In conclusion, we've shown that U' and V' are nonempty, disjoint sets, which are relatively open in E, satisfying $U' \cup V' = E$. Therefore U' and V' separate E, and E is not connected. This is a contradiction, so we can conclude that $E \cup \{\mathbf{a}\}$ must be connected. $\square$

4. Let $\{\mathbf{x}_k\}$ and $\{\mathbf{y}_k\}$ be sequences in $\mathbf{R}^n$. Assume that $\lim_{k\to\infty} \mathbf{y}_k = \mathbf{0}$ and that the sequence $\{\mathbf{x}_k\}$ is bounded. Show that $\lim_{k\to\infty} \mathbf{x}_k \cdot \mathbf{y}_k = 0$.

Proof. Let $\varepsilon > 0$. By the definition of bounded, there exists an $M > 0$ so that $\|\mathbf{x}_k\| \leq M$ for all $k \in \mathbf{N}$. Since we assume $\lim_{k\to\infty} \mathbf{y}_k = \mathbf{0}$, we can choose an $N \in \mathbf{N}$ so that

$$\|\mathbf{y}_k\| = \|\mathbf{y}_k - \mathbf{0}\| < \frac{\varepsilon}{M} \quad \text{for } k \geq N.$$

Using the Cauchy-Schwartz inequality, we then have for all $k \geq N$ that

$$\begin{aligned} |\mathbf{x}_k \cdot \mathbf{y}_k - 0| &= |\mathbf{x}_k \cdot \mathbf{y}_k| \\ &\leq \|\mathbf{x}_k\| \, \|\mathbf{y}_k\| \\ &\leq M \, \|\mathbf{y}_k\| \\ &< M \left(\frac{\varepsilon}{M}\right) = \varepsilon. \end{aligned}$$

We conclude that $\lim_{k\to\infty} \mathbf{x}_k \cdot \mathbf{y}_k = 0$. $\square$