Math 421, Section 2 Test II April 1, 2009

Name:____

Instructions:

- No calculators, books, notes, or electronic devices may be used during the exam.
- You have 50 minutes.
- There are 4 problems, some with multiple parts. Complete all problems, and write all solutions in the space provided on the exam.
- Unless indicated otherwise, you may use any theorem proved in the lecture, or any theorem from single variable differential calculus. When applying a theorem, you should explicitly verify that all necessary hypotheses are met.
- Additional scratch paper is provided at the end of the exam.

Good Luck!

Problem 1 (10 points)	
Problem 2 (15 points)	
Problem 3 (10 points)	
Problem 4 (10 points)	
Total (50 points)	

1. Determine whether each of the following statements is true or false. If a statement is false, provide a counterexample. (Write out the word "true" or "false" completely! No proof or explanation is necessary if you answer "true," and you don't need to prove your proposed counterexample is a counterexample if you answer "false.")

   (a) Let $f : E \subset \mathbf{R}^n \to \mathbf{R}^m$ be a continuous function. If $H \subset \mathbf{R}^m$ is compact, then $f^{-1}(H)$ is compact.

   (b) Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a continuous function. If $U \subset \mathbf{R}^n$ is an open set, then $f(U) \subset \mathbf{R}^m$ is an open set.

   (c) Let $E \subset \mathbf{R}^n$ be a closed set. Let $\{\mathbf{x}_k\}_{k\in\mathbf{N}}$ be a sequence with $\mathbf{x}_k \in E$ for all $k \in \mathbf{N}$, and assume $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}$. Then $\mathbf{x} \in E$.

   (d) Let U, $V \subset \mathbf{R}^n$ be connected sets. Then $U \cap V$ is connected.

2. (a) Define what it means for a set $E \subset \mathbf{R}^n$ to be connected.

(b) Assume that $E \subset \mathbf{R}^n$ is connected. Prove that the closure $\overline{E}$ is also connected.

3. Let $E \subset \mathbf{R}^n$ be a closed set, and assume that $\mathbf{a} \notin E$. Prove that

$$\inf_{\mathbf{x} \in E} \|\mathbf{x} - \mathbf{a}\| > 0.$$

4. Let $U \subset \mathbf{R}^n$ be an open set, and let $\mathbf{a} \in U$. Assume that the function $f : U \setminus \{\mathbf{a}\} \to \mathbf{R}^m$ is uniformly continuous on $U \setminus \{\mathbf{a}\}$. Prove that

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x})$$

exists. (Hint: Use the sequential characterization of limits.)

(scratch paper)

(scratch paper)