1. Determine whether each of the following statements is true or false. If a statement is false, provide a counterexample. (Write out the word "true" or "false" completely!)

   (a) Let $f,\ g : [a,b] \to \mathbf{R}$ be integrable functions. Assume that $f(x) \geq g(x)$ for all $x \in [a,b]$, and that there exists a point $c \in [a,b]$ so that $f(c) > g(c)$. Then $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$.

   Answer: False. Let $[a,b] = [-1,1]$, let $g(x) = 0$, and let

   $$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

   Then $f(x) \geq g(x)$ for all $x \in [-1,1]$, $f(0) > g(0)$, but $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} g(x)\,dx = 0$. (Note that the statement would be true if we were to add the assumption that $f - g$ were continuous; then we could apply the results of problem 4 on page 115 of the textbook.) □

   (b) Let $f : (a,b) \to \mathbf{R}$ be a locally integrable function. Then $|f|$ is a locally integrable function.

   Answer: True. If f is locally integrable, then by definition f is integrable on any interval $[c,d] \subset (a,b)$. Using Theorem 5.22, it follows that $|f|$ is integrable on any interval $[c,d] \subset (a,b)$, which by definition means that $|f|$ is locally integrable. □

   (c) Let $f : [a,b] \to \mathbf{R}$ be a function, and assume that $|f|$ is integrable. Then f is integrable.

   Answer: False. Let $[a,b] = [0,1]$, and let

   $$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ -1 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

   Then f is not integrable on $[0,1]$, but $|f(x)| = 1$ for all $x \in [0,1]$, so $|f|$ is integrable on $[0,1]$. □

   (d) Let $f : [0,\infty) \to \mathbf{R}$ be a locally integrable function. Then f is improperly integrable on $[0,\infty)$ if and only if $\lim_{x\to\infty} f(x) = 0$.

   Answer: False. Let $f(x) = \frac{1}{1+x}$. Then f is continous, and so, locally integrable, $\lim_{x\to\infty} f(x) = 0$, but f is not improperly integrable. □

   (e) Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{w} \in \mathbf{R}^n$, and assume that $\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{w}$, where $\cdot$ denotes the dot product. Then $\mathbf{y} = \mathbf{w}$.

   Answer: False. Let $\mathbf{x} = (1,0)$, $\mathbf{y} = (0,1)$, $\mathbf{w} = (0,-1)$. Then $\mathbf{x} \cdot \mathbf{y} = 0 = \mathbf{x} \cdot \mathbf{w}$. □

2. (a) Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$, and let $f : [a, b] \to \mathbf{R}$ be a bounded function. State the definitions of the upper and lower Riemann sums, $U(f, P)$ and $L(f, P)$, of f over P.

Answer:

$$U(f, P) = \sum_{j=1}^{n} M_j(f)(x_j - x_{j-1})$$

and

$$L(f, P) = \sum_{j=1}^{n} m_j(f)(x_j - x_{j-1})$$

where

$$M_j(f) := \sup_{x \in [x_{j-1}, x_j]} f(x)$$

and

$$m_j(f) := \inf_{x \in [x_{j-1}, x_j]} f(x).$$

□

(b) Define what it means for a function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable.

Answer: A function $f : [a, b] \to \mathbf{R}$ is said to be Riemann integrable if:

(i) f is bounded, and

(ii) for any $\varepsilon > 0$, there exists a partition P of $[a, b]$ so that

$$U(f, P) - L(f, P) < \varepsilon.$$

□

(c) Prove that the function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \sin \frac{1}{x} & \text{if } x > 0 \end{cases}$$

is Riemann integrable on $[0, 1]$.

Proof. First note that f is continous on any interval that does not contain 0, and therefore that f is integrable on any closed bounded interval that does not contain 0.

Let $\varepsilon > 0$. By the previous comment, f is integrable on the interval $[\varepsilon/4, 1]$, and by the definition of integrability, we can choose a partition P_1 of $[\varepsilon/4, 1]$ so that

$$U(f, P_1) - L(f, P_1) < \varepsilon/2.$$

Define a partition P of $[0, 1]$ by $P = \{0\} \cup P_1$. Then

$$\begin{aligned} U(f, P) - L(f, P) &= \left(\sup_{x \in [0, \varepsilon/4]} f(x) - \inf_{x \in [0, \varepsilon/4]} f(x) \right) (\varepsilon/4 - 0) + U(f, P_1) - L(f, P_1) \\ &= (1 - (-1))\, \varepsilon/4 + U(f, P_1) - L(f, P_1) \\ &= \varepsilon/2 + U(f, P_1) - L(f, P_1) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

and therefore f is integrable on $[0, 1]$. □

3. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function. Prove that

$$\lim_{\delta \to 0^+} \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\, dx = f(x_0).$$

Proof 1. In order to prove that

$$\lim_{\delta \to 0^+} \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\, dx = f(x_0).$$

we need to show that for any $\varepsilon > 0$, there exists a δ' so that

$$\delta \in (0, \delta') \qquad \Rightarrow \qquad \left| \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\, dx - f(x_0) \right| < \varepsilon.$$

Let $\varepsilon > 0$. Since f is assumed continous, we can find a $\delta' > 0$ so that

$$|x - x_0| < \delta' \qquad \Rightarrow \qquad |f(x) - f(x_0)| < \varepsilon.$$

For $\delta \in (0, \delta')$, we therefore have that

$$\begin{aligned}
\left| \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\, dx - f(x_0) \right| &= \left| \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\, dx - f(x_0) \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} 1\, dx \right| \\
&= \left| \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\, dx - \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x_0)\, dx \right| \\
&= \left| \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x) - f(x_0)\, dx \right| \\
&\le \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} |f(x) - f(x_0)|\, dx \\
&< \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} \varepsilon\, dx \\
&= \varepsilon.
\end{aligned}$$

Hence $\delta \in (0, \delta')$ implies that

$$\left| \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\, dx - f(x_0) \right| < \varepsilon$$

as required. □

Proof 2. Here we will use the Mean Value Theorem.

Since f is assumed continous, the Mean Value Theorem for integrals implies that for every $\delta > 0$, we can choose an $x_\delta \in [x_0 - \delta, x_0 + \delta]$ so that

$$\frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx = f(x_\delta).$$

Using the squeeze theorem, we see that

$$\lim_{\delta \to 0^+} x_\delta = x_0$$

since

$$x_0 - \delta \le x_\delta \le x_0 + \delta.$$

By continuity of f, we then have that

$$\lim_{\delta \to 0^+} f(x_\delta) = f(x_0),$$

and therefore

$$\lim_{\delta \to 0^+} \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx = \lim_{\delta \to 0^+} f(x_\delta) = f(x_0).$$

□

Proof 3. Here we will use the Fundamental Theorem of Calculus (FTC). (There is some danger of circularity here since we needed to prove a result similar to the one here in order to prove the FTC, but since I did not forbid the use of the FTC in this problem, you were free to use it.)

Define

$$F(x) = \int_c^x f(t)\,dt$$

for any $c \in \mathbf{R}$. Note that since f is assumed to be continous, the FTC implies that F is differentiable, and that $F'(x) = f(x)$.

We then have that

$$\begin{aligned}
\lim_{\delta \to 0^+} \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx &= \lim_{\delta \to 0^+} \frac{1}{2\delta} \left(F(x_0+\delta) - F(x_0-\delta)\right) \\
&= \frac{1}{2} \left(\lim_{\delta \to 0^+} \frac{F(x_0+\delta) - F(x_0)}{\delta} + \lim_{\delta \to 0^+} \frac{F(x_0-\delta) - F(x_0)}{-\delta} \right) \\
&= \frac{1}{2} \left(F'(x_0) + F'(x_0)\right) \\
&= F'(x_0) = f(x_0).
\end{aligned}$$

□

4. Let $f,\ g : [0,\infty) \to \mathbf{R}$ be locally integrable functions, and assume that $g(x) \geq 0$ for all $x \in [0,\infty)$. Assume that the limit
$$L := \lim_{x\to\infty} \frac{f(x)}{g(x)}$$
exists and satisfies $L \in (0,\infty)$. Prove that f is improperly integrable on $[0,\infty)$ if and only if g is improperly integrable on $[0,\infty)$.

Proof. According to the definition of limit, the equation
$$L = \lim_{x\to\infty} \frac{f(x)}{g(x)}$$
means that for any $\varepsilon > 0$, there exists an N, so that $x > N$ implies that
$$\left|\frac{f(x)}{g(x)} - L\right| < \varepsilon.$$
Since $L > 0$ is finite, we can, in particular, find an N, so that $x > N$ implies that
$$\left|\frac{f(x)}{g(x)} - L\right| < \frac{L}{2},$$
or equivalently, so that
$$0 < \frac{L}{2} < \frac{f(x)}{g(x)} < \frac{3L}{2}$$
for all $x > N$. Note that this implies that $g(x)$ is nonzero for all $x > N$, and since $g(x) \geq 0$ for all $x \geq 0$, we have that $g(x) > 0$ for all $x > N$. We can therefore multiply this inequality through by $g(x)$ to find that
$$0 < \frac{L}{2}g(x) < f(x) < \frac{3L}{2}g(x)$$
for all $x > N$.

Now, if g is improperly integrable on $[0,\infty)$, then $\frac{3L}{2}g$ is improperly integrable on $[0,\infty)$ (Theorem 5.42), and therefore $\frac{3L}{2}g$ is improperly integrable on $[N,\infty)$. The inequality
$$0 < f(x) < \frac{3L}{2}g(x) \qquad \text{for all } x > N$$
then allows us to apply the comparison theorem for improper integrals (Theorem 5.43) to conclude that f is improperly integrable on $[N,\infty)$. Since f is assumed locally integrable on $[0,\infty)$, f being improperly integrable on $[N,\infty)$ is equivalent to f being improperly integrable on $[0,\infty)$.

Similarly, if f is improperly integrable on $[0,\infty)$, we use the inequality
$$0 < \frac{L}{2}g(x) < f(x) \qquad \text{for all } x > N$$

with the comparison theorem for improper integrals to conclude that $\frac{L}{2}g$ and g are improperly integrable on $[N, \infty)$, which, with the assumption of local integrability of g, implies that g is improperly integrable on $[0, \infty)$.

$\square$