1. Determine whether each of the following statements is true or false. If a given statement is true, write the word "TRUE" (no explanation or proof is necessary). If a given statement is false, write the word "FALSE" and *provide a concrete counterexample* (you do not need to prove that your proposed counterexample is a counterexample).

   (a) Let $f : [a,b] \to \mathbf{R}$ be an integrable function, and define $F(x) = \int_a^x f(t)\,dt$. If f is continuous at $x_0 \in [a,b]$, then F is differentiable at $x_0 \in [a,b]$.

   True. This follows from the version of the fundamental theorem of calculus stated in class. □

   (b) Let $f : [a,b] \to \mathbf{R}$ be an integrable function, and define $F(x) = \int_a^x f(t)\,dt$. If f is not continuous at $x_0 \in [a,b]$, then F is not differentiable at $x_0 \in [a,b]$.

   False. Let $[a,b] = [0,1]$, and let $f(x) = \begin{cases} 1 & x = \frac{1}{2} \\ 0 & x \in [0,1] \setminus \left\{\frac{1}{2}\right\}. \end{cases}$ Then F is identically zero and therefore differentiable on $[0,1]$ even though f is not continuous at $\frac{1}{2}$.

   Also, if we let $F(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$, then F can be shown to be differentiable on $[0,1]$, and $f := F'$ is bounded on $[0,1]$ and has only one discontinuity. Therefore f is integrable and $F(x) = F(x) - F(0) = \int_0^x f(t)\,dt$. This gives a counterexample because F is differentiable at 0 even though f is not continuous at 0. □

   (c) Assume that $f : [0,\infty) \to \mathbf{R}$ be improperly integrable on $[0,\infty)$. Then f is bounded on $[0,\infty)$.

   False. Let $f(x) = \begin{cases} x & x \in \mathbf{N} \\ 0 & \text{otherwise.} \end{cases}$ Then on any set of the form $[0,r]$ f is bounded, integrable and $\int_0^r f(x)\,dx = 0$ since $f(x) = 0$ except for on a finite subset of $[0,r]$. Therefore $\lim_{r\to\infty} \int_0^r f(x)\,dx = 0$ so f is improperly integrable, but not bounded on $[0,\infty)$.

   Note that one can also construct continuous counterexamples (for instance, it's not too hard to construct a continuous function f having a "bump" near each natural number n of height n and total area 2^{-n}). □

   (d) Let $f : [a,b] \to \mathbf{R}$ be an integrable function. Define $g : [a,b] \to \mathbf{R}$ by $g(x) = (f(x))^5$. Then g is integrable on $[a,b]$.

   True. Products of integrable functions are integrable (Corollary 5.23), and therefore, by induction, any positive integer power of an integrable function is integrable. □

2. (a) Let $f : [a, b] \to \mathbf{R}$ be a bounded function.

i. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Give the definition of the *lower Riemann sum* of f over P.

Definition. The lower Riemann sum $L(f, P)$ of f over P is defined by

$$L(f, P) = \sum_{j=1}^{n} m_j(f)(x_j - x_{j-1})$$

where $m_j(f) = \inf f([x_{j-1}, x_j])$. $\square$

ii. Give the definition of *lower integral* of f on $[a, b]$.

Definition. The lower integral $(L) \int_a^b f(x)\, dx$ of f on $[a, b]$ is defined by

$$\sup \{L(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

$\square$

(b) Let $f : [a, b] \to \mathbf{R}$ be a bounded function, and assume that the lower integral of f on $[a, b]$ is positive. Show that there is exists a (nondegenerate) interval $[c, d] \subseteq [a, b]$ so that $f(x) > 0$ for all $x \in [c, d]$.

Proof. Denote $I = (L) \int_a^b f(x)\, dx > 0$. By the definition of lower integral and the approximation property of suprema, there is a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ so that

$$L(f, P) \geq \frac{I}{2} > 0.$$

We claim this implies that for at least one value of $j \in \{1, \ldots, n\}$ that $m_j(f) > 0$. Indeed, if $m_j(f) \leq 0$ for every $j \in \{1, \ldots, n\}$ then it follows immediately from the definition of $L(f, P)$ above that $L(f, P) \leq 0$ which would be a contradiction. Letting j' then be a value where $m_{j'}(f) > 0$, the definition of infimum implies for all $x \in [x_{j'-1}, x_{j'}]$ that $f(x) \geq \inf f([x_{j'-1}, x_{j'}]) =: m_{j'}(f) > 0$. $\square$

3. Let $f : [a,b] \to \mathbf{R}$ be an integrable function. Let $g : \mathbf{R} \to \mathbf{R}$ be a function, and assume there exists a real number $K > 0$ so that

$$|g(x) - g(y)| \le K\,|x - y|$$

for all x, $y \in \mathbf{R}$. Show that the composition $g \circ f : [a,b] \to \mathbf{R}$ is integrable.[1]

Proof. We first observe that since f is integrable, f must be bounded so there exists an $M > 0$ so that $f(x) \le M$ for all $x \in [a,b]$. We then find that for any $x \in [a,b]$

$$\begin{aligned}
|(g \circ f)(x)| &\le |g(f(x)) - g(f(a))| + |(g \circ f)(a)| \\
&\le K\,|f(x) - f(a)| + |(g \circ f)(a)| \\
&\le K\,(|f(x)| + |f(a)|) + |(g \circ f)(a)| \\
&\le 2KM + |(g \circ f)(a)|\,.
\end{aligned}$$

Therefore $g \circ f$ is bounded on $[a,b]$.[2]

We next claim that for any $A \subset [a,b]$ that

$$\sup_{x\in A}(g \circ f)(x) - \inf_{x\in A}(g \circ f)(x) \le K\left(\sup_{x\in A} f(x) - \inf_{x\in A} f(x)\right). \tag{1}$$

To see this, we first note our assumption about g imply that for any x, $y \in A$,

$$(g\circ f)(x) - (g\circ f)(y) \le |(g \circ f)(x) - (g \circ f)(y)| = |g(f(x)) - g(f(y))| \le K\,|f(x) - f(y)|\,. \tag{2}$$

If follows from the definitions of sup and inf that we can conclude that

$$-\left(\sup_{x\in A} f(x) - \inf_{x\in A} f(x)\right) \le f(x) - f(y) \le \sup_{x\in A} f(x) - \inf_{x\in A} f(x) \text{ for all } x,\ y \in A$$

which is equivalent to

$$|f(x) - f(y)| \le \sup_{x\in A} f(x) - \inf_{x\in A} f(x) \qquad \text{for all } x,\ y \in A. \tag{3}$$

Combining (2) and (3) gives us

$$(g \circ f)(x) - (g \circ f)(y) \le K\left(\sup_{x\in A} f(x) - \inf_{x\in A} f(x)\right) \qquad \text{for all } x,\ y \in A. \tag{4}$$

[1] N.B. It is not true in general that compositions of Riemann integrable functions are Riemann integrable.

[2] An alternate argument here could use the fact that the the condition

$$|g(x) - g(y)| \le K\,|x - y| \qquad \text{for all } x,\ y \in \mathbf{R}$$

implies that g is continuous. Since f must be bounded, we know that $f([a,b]) \subset [-M, M]$ for some $M > 0$. Then since g is continuous, the extreme value theorem tells us that g is bounded on $[-M, M]$, and hence g is bounded on $f([a,b])$. (Note that g need not be bounded on all of $\mathbf{R}$.)

Therefore

$$\sup_{x\in A}(g\circ f)(x) - \inf_{x\in A}(g\circ f)(x) = \sup_{x,y\in A}\left[(g\circ f)(x) - (g\circ f)(y)\right] \le K\left(\sup_{x\in A} f(x) - \inf_{x\in A} f(x)\right)$$

which gives us (1)[3]

Let $\varepsilon > 0$. Since f is assumed to be integrable on $[a,b]$, we can choose a partition $P = \{x_j\}_{j=0}^n$ of $[a,b]$ so that $U(f,P) - L(f,P) < \varepsilon/K$. We then apply (1) to conclude that

$$M_j(g\circ f) - m_j(g\circ f) \le K\left(M_j(f) - m_j(f)\right)$$

and hence

$$\begin{aligned} U(g\circ f,P) - L(g\circ f,P) &= \sum_{j=1}^n \left(M_j(g\circ f) - m_j(g\circ f)\right)(x_j - x_{j-1}) \\ &\le \sum_{j=1}^n K\left(M_j(f) - m_j(f)\right)(x_j - x_{j-1}) \\ &= K\left(U(f,P) - L(f,P)\right) < K\frac{\varepsilon}{K} = \varepsilon. \end{aligned}$$

Therefore $g\circ f$ is integrable on $[a,b]$. $\square$

[3] There are other ways one could argue (4) implies (1). One could argue as in the proof of Theorem 5.22 in the textbook.

Another way one might argue is to use the approximation property of sup and inf to choose sequences x_k, and $y_k \in A$ satisfying $\lim_{k\to\infty}(g\circ f)(x_k) = \sup_{x\in A}(g\circ f)(A)$ and $\lim_{k\to\infty}(g\circ f)(y_k) = \inf_{x\in A}(g\circ f)(x)$. It would then follow from (4) that

$$(g\circ f)(x_k) - (g\circ f)(y_k) \le K\left(\sup_A f - \inf_A f\right)$$

and therefore

$$\sup_A g\circ f - \inf_A g\circ f = \lim_{k\to\infty}\left((g\circ f)(x_k) - (g\circ f)(y_k)\right) \le K\left(\sup_A f - \inf_A f\right).$$

4. Let $f : [0, \infty) \to \mathbf{R}$ and $g : [0, \infty) \to \mathbf{R}$ be nonnegative functions (i.e. $g(x) \geq 0$ and $f(x) \geq 0$ for all $x \in [0, \infty)$). Assume that f and g are locally integrable on $[0, \infty)$, and that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

Show that if g is improperly integrable on $[0, \infty)$ then so is f.

Proof. By definition of $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, we can find an $R \geq 0$ so that

$$\left| \frac{f(x)}{g(x)} \right| = \left| \frac{f(x)}{g(x)} - 0 \right| < 1 \qquad \text{for all } x \geq R.$$

This implies that $|f(x)| \leq |g(x)|$ for all $x \geq R$, which since f and g are assumed to be nonnegative implies that

$$0 \leq f(x) \leq g(x) \qquad \text{for all } x \geq R.$$

If g is improperly integrable on $[0, \infty)$, then g is improperly integrable on $[R, \infty)$, so by the comparison theorem for improper integrals (Theorem 5.43), we can conclude that f is improperly integrable on $[R, \infty)$, and thus f is improperly integrable on $[0, \infty)$. $\square$