MATH 421, SECTION 2 TEST I FEBRUARY 18, 2009

Name:____

Instructions:

- No calculators, books, notes, or electronic devices may be used during the exam.
- You have 50 minutes.
- There are 4 problems, some with multiple parts. Complete all problems, and write all solutions in the space provided on the exam.
- Unless indicated otherwise, you may use any theorem proved in the lecture, or any theorem from single variable differential calculus. When applying a theorem, you should explicitly verify that all necessary hypotheses are met.
- Additional scratch paper is provided at the end of the exam.

Good Luck!

Problem 1 (15 points)	
Problem 2 (15 points)	
Problem 3 (10 points)	
Problem 4 (10 points)	
Total (50 points)	

1. Determine whether each of the following statements is true or false. If a statement is false, provide a counterexample. (Write out the word "true" or "false" completely!)

   (a) Let f, $g : [a, b] \to \mathbf{R}$ be integrable functions. Assume that $f(x) \geq g(x)$ for all $x \in [a, b]$, and that there exists a point $c \in [a, b]$ so that $f(c) > g(c)$. Then $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$.

   (b) Let $f : (a, b) \to \mathbf{R}$ be a locally integrable function. Then $|f|$ is a locally integrable function.

   (c) Let $f : [a, b] \to \mathbf{R}$ be a function, and assume that $|f|$ is integrable. Then f is integrable.

   (d) Let $f : [0, \infty) \to \mathbf{R}$ be a locally integrable function. Then f is improperly integrable on $[0, \infty)$ if and only if $\lim_{x\to\infty} f(x) = 0$.

   (e) Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{w} \in \mathbf{R}^n$, and assume that $\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{w}$, where $\cdot$ denotes the dot product. Then $\mathbf{y} = \mathbf{w}$.

2. (a) Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$, and let $f : [a, b] \to \mathbf{R}$ be a bounded function. State the definitions of the upper and lower Riemann sums, $U(f, P)$ and $L(f, P)$, of f over P.

(b) Define what it means for a function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable.

(c) Prove that the function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \sin \frac{1}{x} & \text{if } x > 0 \end{cases}$$

is Riemann integrable on $[0, 1]$.

3. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function. Prove that

$$\lim_{\delta \to 0^+} \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} f(x)\, dx = f(x_0).$$

4. Let $f,\ g : [0, \infty) \to \mathbf{R}$ be locally integrable functions, and assume that $g(x) \geq 0$ for all $x \in [0, \infty)$. Assume that the limit
$$L := \lim_{x\to\infty} \frac{f(x)}{g(x)}$$
exists and satisfies $L \in (0, \infty)$. Prove that f is improperly integrable on $[0, \infty)$ if and only if g is improperly integrable on $[0, \infty)$.

(scratch paper)

(scratch paper)