Math 421, Section 2 Final Exam May 8, 2009

Name:________________________________

Instructions:

- No calculators, books, notes, or electronic devices may be used during the exam.
- You have 2 hours.
- There are 7 problems, some with multiple parts. Complete all problems, and write all solutions in the space provided on the exam.
- Unless indicated otherwise, you may use any theorem proved in the lecture, or any theorem from single variable differential calculus. When applying a theorem, you should explicitly verify that all necessary hypotheses are met.
- Additional scratch paper is provided at the end of the exam.

Good Luck!

Problem 1 (15 points)	
Problem 2 (10 points)	
Problem 3 (15 points)	
Problem 4 (10 points)	
Problem 5 (15 points)	
Problem 6 (10 points)	
Problem 7 (10 points)	
Total (85 points)	

1. Determine whether each of the following statements is true or false. If a statement is false, provide a counterexample. (Write out the word "true" or "false" completely! No proof or explanation is necessary if you answer "true," and you don't need to prove your proposed counterexample is a counterexample if you answer "false.")

   (a) Let $f,\, g : [a, b] \to \mathbf{R}$ be integrable functions. Assume that $f(x) \geq g(x)$ for all $x \in [a, b]$, and that there exists a point $c \in [a, b]$ so that $f(c) > g(c)$. Then $\int_a^b f(x)\, dx > \int_a^b g(x)\, dx$.

   (b) Let $f : (a, b) \to \mathbf{R}$ be a locally integrable function. Then $|f|$ is a locally integrable function.

   (c) Let $E \subset \mathbf{R}^n$ be a connected set. Then the interior E^o of E is connected.

(d) Let $E \subset \mathbf{R}^n$ be a closed set. Let $\{\mathbf{x}_k\}_{k\in\mathbf{N}}$ be a sequence with $\mathbf{x}_k \in E$ for all $k \in \mathbf{N}$, and assume $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}$. Then $\mathbf{x} \in E$.

(e) Let $\{U_k\}_{k=1}^{\infty}$ be a collection of open subsets of $\mathbf{R}^n$. Then the intersection $\cap_{k=1}^{\infty} U_k$ is open.

(f) Consider a function $f : \mathbf{R}^2 \to \mathbf{R}$, and assume that $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist. Then f is differentiable at $(0,0)$.

(g) Let $V \subset \mathbf{R}^n$ be an open, connected, set, and let $f : V \to \mathbf{R}^m$ a function. Assume that $\frac{\partial f}{\partial x_j}(\mathbf{x}) = \mathbf{0}$ for all $j \in \{1,\dots,n\}$, and all $\mathbf{x} \in V$. Then f is constant on V.

2. Let $f : [a, b] \to \mathbf{R}$ be a bounded function, and assume that there is a $c \in (a, b)$ so that f is continuous on $[a, c)$ and on $(c, b]$. Prove that f is integrable on $[a, b]$.

3. (a) Define what it means for a subset U of $\mathbf{R}^n$ to be an *open set.*

(b) Let $U \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^n$ be open sets. Using only the definition of open set (and not any theorems proved in lecture or the textbook), prove that the intersection $U \cap V$ is an open set.

4. Let E be a subset of $\mathbf{R}^n$. A point $\mathbf{a} \in \mathbf{R}^n$ is called a *cluster point* of E if for every $r > 0$ the set $E \cap B_r(\mathbf{a})$ contains infinitely many points. Prove that every bounded infinite subset of $\mathbf{R}^n$ has at least one cluster point.

5. (a) Let $V \subset \mathbf{R}^n$ be an open set. Define what it means for a function $f : V \to \mathbf{R}^m$ to be differentiable at a point $\mathbf{a} \in V$.

(b) Define a function $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x, y) = \begin{cases} \frac{x^3+x^2+y^3+y^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0). \end{cases}$$

At what points is f differentiable? (Support your answer!)

6. Let $V \subset \mathbf{R}^n$ be an open set, and let $f : V \to \mathbf{R}^m$ be a function. Define the directional derivative $D_{\mathbf{v}} f(\mathbf{a})$ of f at $\mathbf{a} \in V$ along a vector $\mathbf{v} \in \mathbf{R}^n$ by

$$D_{\mathbf{v}} f(\mathbf{a}) := \lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a})}{t} = \left. \frac{d}{dt} \right|_{t=0} f(\mathbf{a} + t\mathbf{v}).$$

Prove that if f is differentiable at $\mathbf{a}$, then

$$D_{\mathbf{v}} f(\mathbf{a}) = Df(\mathbf{a})\mathbf{v}$$

for all $\mathbf{v} \in \mathbf{R}^n$.

7. Show that there exist an $r > 0$ and continuously differentiable functions

$$u(s,t) : B_r((0,0)) \subset \mathbf{R}^2 \to \mathbf{R}$$
$$v(s,t) : B_r((0,0)) \subset \mathbf{R}^2 \to \mathbf{R}$$

so that

$$u(0,0) = 2, \qquad v(0,0) = 0,$$

and so that

$$s\cos(u^2+5) + \sin v - t = 0$$
$$(u^3 - u^2)e^t - sv^2 + stuv + s^3 t = 4$$

for all $(s,t) \in B_r((0,0))$.

(scratch paper)

(scratch paper)

(scratch paper)