Thomas Calculus; 12^{th} Edition: The Power Rule

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The following paragraph appears at the bottom of page 116 of Thomas Calculus, 12^{th} Edition.

> The Power Rule is actually valid for all real numbers n. We have seen examples for negative integers and fractional powers, but n could be an irrational number as well. to apply the Power Rule, we subtract 1 from the original exponent n and multiply the result by n. Here we state the general version of the rule, but postpone its proof until Chapter 7.

The definition of the power function x^a for a an irrational number isn't defined until Chapter 7 after the introduction of the exponential function, e^x. So it shouldn't be mentioned at this point in the text. Establishing the Power Rule for functions of the form x^r where r is rational can be done using the Chain Rule, which is covered in the current chapter. The first step is to prove the rule in the special case $x^{\frac{1}{n}}$ where $n \in \mathbb{N}$. To do so, use the same technique used immediately preceding the paragraph mentioned above to verify the Power Rule for x^n. Specifically for $f(x) = x^{\frac{1}{n}}$

$$
\begin{aligned}
f'(x) = \lim_{z \to x} \frac{x^{\frac{1}{n}} - z^{\frac{1}{n}}}{x - z} &= \lim_{z \to x} \frac{x^{\frac{1}{n}} - z^{\frac{1}{n}}}{(x^{\frac{1}{n}})^n - (z^{\frac{1}{n}})^n} \\
&= \lim_{z \to x} \frac{x^{\frac{1}{n}} - z^{\frac{1}{n}}}{\left((x^{\frac{1}{n}}) - (z^{\frac{1}{n}})\right)\left(\sum_{k=0}^{n-1} (x^{\frac{1}{n}})^{n-1-k} (z^{\frac{1}{n}})^k\right)} \\
&= \frac{1}{n(x^{\frac{1}{n}})^{n-1}} = \frac{1}{n} x^{\frac{1}{n}-1}
\end{aligned}
$$

Next for $n \in \mathbb{N}$ apply the Chain Rule to $x^{-n} = (x^{-1})^n$ to extend the Power Rule to x^n for $n \in \mathbb{Z}$. Finally for $r = \frac{m}{n}$ where $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ apply the Chain Rule to $x^r = (x^{\frac{1}{n}})^m$ to prove the Power Rule $\frac{d}{dx} x^r = r x^{r-1}$.