Notes on Power Series

The most general kind of power series is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n,$$

where c is a real number called the *center* of the series and a_n is a real number called the *n-th coefficient.* For a given power series we are interested in:

- finding those x's for which the series converges,
- determining how the series converges for these x's, and (if we're lucky)
- finding, for these x's, the sum of the series.

Since we can reduce a power series with center at c to one with center at 0 just by making the change of variable $y = x - c$, which recasts the original series as $\sum_{n=0}^{\infty} a_n y^n$. From now on we'll consider only consider power series with center at 0, i.e. those of the form

$$\text{(PS)} \qquad \sum_{n=0}^{\infty} a_n x^n \ ,$$

Here are five examples that illustrate most everything that can happen; the details are left to you as exercises.

Example 1. $\displaystyle\sum_{n=0}^{\infty} x^n.$

This is the *Geometric Series,* which we know converges absolutely for each x in the open interval $(-1, 1)$, and diverges for each x outside this interval. We showed in class last Friday that, although it does not converge uniformly on this interval, it *does* converge uniformly on each compact subset—i.e., uniformly on each closed subinterval $[-a, a]$ for $0 < a < 1$.

Example 2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} x^n.$

This series converges for $x = -1$ by the Alternating Series Test, and *diverges* at $x = 1$ (where it becomes the famous harmonic series). In between it converges absolutely (by "comparison" with the geometric series), and even uniformly on compact subsets of $(-1, 1)$ (by the Weierstrass M-test with, say, $M_n = a^n$ for $0 < a < 1$). This series diverges for each x outside $[-1, 1)$ (since for such x's, $\lim_{n\to\infty} a^n x^n$ does not exist).

Example 3. $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$.

This series converges absolutely and uniformly on the closed interval $[-1, 1]$ (Weierstrass M-test with $M_n = 1/n^2$), and diverges for each x outside this interval—exercise!.

Example 4. $\sum_{n=1}^{\infty} n^n x^n$.

This series converges *only at the origin,* (n-th term test, since $\lim_{n\to\infty} n^n x^n$ does not even exist for any $x \neq 0$.)

Example 5. $\sum_{n=0}^{\infty} \frac{1}{n^n} x^n$.

This series converges absolutely for every $x \in \mathbb{R}$, and the convergence is *uniform* on every compact subset of $\mathbb{R}$ (Weierstrass M-test with $M_n = a^n/n^n$ for each $0 < a < \infty$).

So, except for Example 4, there is an open interval on which our power series converges absolutely, on compact subsets of which the series converges uniformly, and outside of whose closure the series diverges.

The main theorem about convergence of power series asserts that this is no accident:

The Fundamental Theorem of Power Series (FTPS). *For each power series PS there are only three possibilities:*

(a) *The series converges only for* $x = 0$.

(b) *The series converges absolutely for all* $x \in \mathbb{R}$, *uniformly on compact subsets of* $\mathbb{R}$.

(c) *There exists* $R > 0$ *(called the "radius of convergence" of the series) such that (PS) converges absolutely at each point of the open interval* $(-R, R)$, *uniformly on compact subsets of that interval, and diverges at each point outside the closed interval* $[-R, R]$.

The proof of this theorem depends critically on the following result, which is an amalgamation of Theorems 6.5.1 and 6.5.2 of our textbook.

Fundamental Lemma (FL). *If PS converges at a point* $x_0 \neq 0$ *then it converges absolutely for every* x *with* $|x| < |x_0|$. *If the convergence at* x_0 *is* absolute *then PS converges* uniformly *on* $[-|x_0|, |x_0|]$.

Remark. Note that convergence at x_0 need not imply convergence at $-x_0$, as Example 2 shows (the series converges conditionally at $x_0 = -1$, but diverges at $x_0 = 1$). However the second statement of FL implies the easily proved fact that if PS converges *absolutely* at x_0, then it does the same at $-x_0$.

Proof of FL. (i) If PS converges absolutely at x_0 then uniform convergence on $[-|x_0|, |x_0|]$ follows from the Weierstrass M-test with $M_n = |a_n\, x_0^n|$. I leave the details to you.

(ii) If the convergence at x_0 is not absolute, we at least know that the numerical sequence $(a_n\, x_0^n)$ converges to zero (n-th term test), so it is, in particular, *bounded:* There exists $M > 0$ such that

$$|a_n\, x_0^n| \le M \qquad (n = 0, 1, 2, \ldots).$$

Thus for any x in $(-|x_0|, |x_0|)$ we have for each n:

$$|a_n\, x^n| = |a_n||x|^n = |a_n|\, |x_0|^n \frac{|x|^n}{|x_0|^n} = |a_n\, x_0^n| \left(\frac{|x|}{|x_0|}\right)^n \le M\, r^n \ ,$$

where $0 < r \stackrel{\text{def}}{=} \frac{|x|}{|x_0|} < 1$. Thus the absolute convergence of PS at x follows from the Comparison test and the convergence of the positive-term geometric series $\sum_n M\, r^n$. For each a with $0 < a < |x_0|$ the uniform convergence of PS on $[-a, a]$ follows, by (i) above, from its absolute convergence at a. ///

Proof of FTPS. Given a series PS, let

$$C = \{r \ge 0 : \text{PS converges for} |x| = r\},$$

Note that $0 \in C$.

(i) If $C = \{0\}$ then it's clear that PS converges only for $x = 0$.

(ii) If C is unbounded then given $x \in \mathbb{R}$ we can find $r \in C$ with $r > |x|$. Since PS converges at r it must converge absolutely at x. Thus PS converges absolutely at $x \in \mathbb{R}$. By the second part of FL, PS therefore converges uniformly on each interval $[-x, x]$ $(x > 0)$, and therefore uniformly on every compact subset of $\mathbb{R}$.

(iii) The one remaining case is: *C is bounded and* $\ne \{0\}$. In this case $0 < R \stackrel{\text{def}}{=} \sup C < \infty$. We will show that R is the radius of convergence of PS.

Suppose $|x| < R$ Then since R is the *least* upper bound of C, $|x|$ is not an u.b. for C, so there is $\rho \in C$ with $|x| < \rho$. PS converges at ρ (definition of C), so by the first part of FL, it converges absolutely at x, and hence unformly on $[-|x|, |x|]$. Thus PS converges absolutely on $(-R, R)$, and uniformly on compact subsets of that interval.

If, on the other hand, $|x| > R$ then PS must diverge at x. For if not, then it converges there, hence $|x| \in C$, and so R cannot be an upper bound for C—a contradiction. ///

Remark. Examples 1–3 show that anything can happen at the endpoints of $(-R, R)$: convergence at neither one, convergence at one but not the other, or convergence at both.

Terminology. If $R > 0$ we call $(-R, R)$ the *open interval of absolute convergence.* Here we allow $R = \infty$ in case PS converges for every x.

Because of uniform convergence on compact subsets of the open interval of absolute convergence, each power series converges to a continuous function there. However even more is true: if you term-by-term differentiate the series, the new series still converges to a continuous function! This is the content of:

Lemma on Differentiated Power Series (LDPS).[*] *Suppose PS has radius of convergence $R > 0$. Then the "termwise differentiated series"*

$$(DPS) \qquad \sum_{n=1}^{\infty} n a_n x^{n-1}$$

converges absolutely on $(-R, R)$ and uniformly on compact subsets of that interval.

Proof. Fix $x \in (-R, R)$ and choose b with $|x| < b < R$. Then

$$(1) \qquad |n a_n x^{n-1}| = n|a_n| b^{n-1} \left(\frac{|x|}{b} \right)^{n-1}.$$

Now $0 \le |x|/b < 1$, so $n(|x|/b)^n \to 0$ as $n \to \infty$ (Exercise 2 below), so in particular the sequence $((|x|/b)^n)$ is bounded, say by $M > 0$. This and (1) above imply that

$$|n a_n x^{n-1}| \le M\,|a_n|\,b^n \qquad (n \in \mathbb{N}),$$

so DPS converges absolutely at x by comparison with the series $\sum_n M\,|a_n|\,b^n$, which converges because $b \in (-R, R)$.

By the first part of FL, DPS therefore converges uniformly on every interval $[-x, x]$ for $|x| < R$, hence uniformly on every compact subset of $(-R, R)$. ///

On PS's the open interval of absolute convergence $(-R, R)$, the uniform convergence of PS on compact subsets guarantees that PS converges to a function f continuous on I, and similarly the termwise differentiated series DPS also converges to a continuous function. What could this new function be? The obvious guess is that f is differentiable on $(-R, R)$ and that DPS converges to f'. In fact FTPS, LDPS, and Theorem 6.3.1 of our text show that this is exactly what happens! We state this result officially as:

Theorem on Differentiation of Power Series.[†] *Let I denote the open interval of absolute convergence of* PS*. Then on I PS converges absolutely, and uniformly on compact subsets to a function f that is differentiable on I, and* DPS *converges in the same fashion to f'.*

Note that we can now repeat the procedure with DPS in place of PS and f' in place of f. In fact, we can do this *ad infinitum:*

Corollary: Infinite differentiability of power series. *On its open interval of absolute convergence,* PS *converges to a function that is infinitely differentiable on that interval. Moreover, for each $n \in N$ the n-th deriviative of f is computed by termwise differentiating* PS *n times.*

[*] Theorem 6.5.6 of our text

[†] Theorem 6.5.7 of our text

Here "f is infinitely differentiable on I" means that for each $x \in I$ and each $n \in \mathbb{N}$, the n-th derivative $f^{(n)}(x)$ exists.

Corollary.[‡] *Suppose PS has radius of convergence $R > 0$, and that f is its sum on $(-R, R)$. Then*

$$a_n = \frac{f^{(n)}(0)}{n!} \qquad n = 0, 1, 2, \dots \ .^{\S}$$

Proof. Set $x = 0$ in PS to get $a_0 = f(0)$. Set $x = 0$ in DPS to get $f'(0) = a_1$, and continue. I leave the rest to you. ///

Taylor Series. If f a function that is infinitely differentiable on an interval $(-R, R)$ then we can always write down the so-called *Taylor series* for f (with center at 0):

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n ,$$

and ask whether or not it converges to f.

If f is already given as the sum of a power series that converges on $(-R, R)$, then the last Corollary says that this is indeed the case. In general, however, it is not, as shown by exercises 6.6.9–6.6.11. In these exercises it's shown that you can have functions f, vanishing only at the origin, that are infinitely differentiable on $\mathbb{R}$, but all of whose derivatives vanish at the origin. The Taylor series for such an f has all coefficients zero, hence converges to zero everywhere, but, except for the origin, does not converge to f.

So the big question is: How can you tell when the Taylor series of f converges to f?

The answer is provided by:

Lagrange's Remainder Theorem (LRT).[¶] *Suppose f is infinitely differentiable on $(-R, R)$. For $n = 0, 1, 2, \ \dots$ let*

$$S_n[f](x) = \sum_{k=0}^{m} \frac{f^{(k)}(0)}{n!} x^k ,$$

the n-th partial sum of the Taylor series of f. Then for each x in $(-R, R)$ there exists a point c between 0 and x such that

(2) $$f(x) = S_n[f](x) + \frac{f^{(n+1)}(c)}{n!} x^{n+1} .$$

Remarks. (a) The case $n = 0$ of LRT is just the Mean Value Theorem!

(b) The "error term" in LRT—the difference between $f(x)$ and $S[f]_n(x)$, looks just like the $n+1$-st term of the Taylor series, except that instead of being evaluated at 0, the derivative is evaluated at the "unknown" point c.

[‡]Exercise 6.6.3 of our text.

[§]Here $0! = 1$, and $f^{(0)} = f$.

[¶]Theorem 6.6.1 of our text.

Example. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1!} x^{2n+1}$, where the series converges at each point of $\mathbb{R}$, uniformly on compact subsets.

Proof. You did this in Calculus. To review: you showed that the series on the right is the Taylor series for $\sin x$. Since each derivative of the sine function is either $\pm \sin x$ or $\pm \cos x$, the "error term" in LRT, for $f(x) = \sin x$, can be estimated as follows:

$$\left| \frac{f^{(n+1)}(c)}{n!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{n!}$$

Now the right-hand side of this inequality converges to 0 for every $x \in \mathbb{R}$ (Exercise 3 below), so $S_n[f](x) \to \sin x$ for every $x \in \mathbb{R}$, as desired. ///

To prove the LRT it turns out to be easier to prove something more general—a "generalized Generalized Mean Value Theorem:"

Generalized Generalized Mean Value Theorem (GGMVT). *Suppose f and g are infinitely differentiable functions on $(-R, R)$ and n is a non-negative integer. Then for every $x \in (-R, R)$ there exists $c = c(x, n)$ between 0 and x such that:*

$$\{f(x) - S_n[f](x)\}\, g^{(n+1)}(c) = \{g(x) - S_n[g](x)\}\, f^{(n+1)}(c) \ . \tag{3}$$

How GGMVT proves LRT. Set $g(x) = x^{n+1}$, so that $g^{(k)}(0) = 0$ for $k = 0, 1, \ldots n$, and $g^{(n+1)} \equiv (n+1)!$. Upon substituting these observations into (3) we obtain

$$\{f(x) - S_n[f](x)\}\, (n+1)! = x^{n+1} f^{(n+1)}(c),$$

which is just (2) in disguise. ///

Proof of GGMVT. Fix $x \in (-R, R)$ and, for t between 0 and x, define:

$$\begin{aligned} F(t) &= f(t) + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k, \quad \text{and} \\ G(t) &= g(t) + \sum_{k=1}^{n} \frac{g^{(k)}(t)}{k!}(x-t)^k \end{aligned}$$

Apply the Generalized Mean Value Theorem (Theorem 5.3.5 of our text) to F and G on the interval of points t between 0 and x to obtain a point c between 0 and x such that

$$F'(c)\{G(x) - G(0)\} = G'(c)\{F(x) - F(0)\} \ .$$

Now $F(x) = f(x)$ and $F(0) = S_n[f](x)$. Similarly for G. Thus the last equation asserts that

$$F'(c)\,\{g(x) - S_n[g](x)\} = G'(c)\,\{f(x) - S_n[f](x)\} \ . \tag{4}$$

We finish the proof by computing $F'(c)$ and $G'(c)$. Using the Chain Rule for differentiation:

$$F'(t) = f'(t) + \sum_{k=1}^{n} \left\{ \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} \right\}$$

The sum on the right telescopes to $\dfrac{f^{(n+1)}(t)}{n!}(x-t)^n - f'(t)$, hence

$$F'(t) = \frac{f^{(n+1)}(t)}{n!}(x-t)^n \ ,$$

with a similar formula for $G'(t)$. Upon evaluating these formulas at $t = c$, substituting the results into (4) and cancelling common factors on each side of the resulting equation, we obtain (3). ///

Exercises

1. Supply the details for the statements made in Examples 1, 2, 4, and 5.
2. Prove that $\lim_{n\to\infty} n\,x^n = 0$ for every $x \in (-1, 1)$.
3. Prove that $\dfrac{x^n}{n!} = 0$ for every $x \in \mathbb{R}$.
4. (Extra Credit) Exercises 6.6.9–6.6.11.