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On the Toeplitzness of Composition Operators

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For Peter Duren, in celebration of his seventieth birthday.

Composition operators on H^2 cannot—except trivially—be Toeplitz, or even "Toeplitz plus compact." However there are natural ways in which they can be "asymptotically Toeplitz." We show here that the study of such phenomena leads to surprising results and interesting open problems.

Introduction

Each holomorphic function φ that takes the open unit disc $\mathbb{U}$ into itself induces a linear *composition operator* C_φ on the space $\mathrm{Hol}\,(\mathbb{U})$ of all functions holomorphic on $\mathbb{U}$ in the following way:

$$C_\varphi f = f \circ \varphi \qquad\qquad (f \in \mathrm{Hol}\,(\mathbb{U})).$$

A consequence of Littlewood's Subordination Principle [10] is the (not at all obvious) fact that every composition operator restricts to a bounded operator on the Hardy space H^2 (see also [5, Theorem 1.7, page 10] or [16, pp. 13–15]), and this in turn has inspired a lively enterprise connecting complex function theory with operator theory, the goal being to understand how properties C_φ are related to those of φ (see [3, 7, 16] for more on this).

The work we will describe here has its roots in the paper [1] of Barría and Halmos, who introduced the notion of "asymptotic Toeplitz operator." One can think of a Toeplitz operator on H^2 as a bounded linear operator whose matrix, relative to the orthonormal basis of monomials $\{z^n : n \geq 0\}$, has constant diagonals. Such operators T can be characterized by the equation

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$S^*TS = T$, where S is the forward shift on H^2 ($Sz^n = z^{n+1}$, $n \geq 0$)), and S^*, its Hilbert space adjoint, is easily seen to be the *backward shift* ($S^*z^n = z^{n-1}$ if $n > 0$, and $= 0$ if $n = 0$). The point here, as noted by Barría and Halmos, is that composing with S on the right erases the first column of your matrix, while composing with S^* on the left erases the first row. Thus passing from T to S^*TS moves the matrix of T one step up the main diagonal, and so leaves the matrix unchanged if and only each diagonal is constant.

Barría and Halmos called an operator T on H^2 *asymptotically Toeplitz* if the sequence of operators $\{S^{*n}TS^n\}$ converges strongly (i.e. pointwise) on H^2. Feintuch [6] pointed out that one needn't rule out either weak or norm (i.e. uniform) operator convergence; hence there are actually *three* different kinds of "asymptotic toeplitzness:" weak, strong (the original one), and uniform.

This paper attempts to characterize the composition operators on H^2 that possess each of these asymptotic notions of toeplitzness. This goal is achieved in two cases: (1) uniform asymptotic toeplitzness (thanks to Feintuch's characterization of such operators as just the compact perturbations of the Toeplitz operators), and (2) the case $\varphi(0) = 0$. For weak and strong asymptotic toeplitzness, however, investigation of the case $\varphi(0) \neq 0$ leads to surprises and interesting open problems.

Here is a more detailed outline of what is to follow. In Section 1 we show that a composition operator can be uniformly asymptotically Toeplitz only trivially, that is, only if it is either compact or the identity. Section 2, by contrast, raises the possibility that *all* composition operators, except those induced by rotations, may be *weakly* asymptotically Toeplitz. While we cannot as yet prove this, we are able to show that for every composition operator C_φ the *arithmetic means* of the sequence $\{S^{*n}C_\varphi S^n\}$ converge in the weak operator topology (Theorem 2.2).

Strong asymptotic toeplitzness, the original concept introduced by Barría and Halmos, is the subject of Section 3. Here initial results, proved under the hypothesis $\varphi(0) = 0$, suggest that the requirement "$|\varphi| < 1$ a.e. on $\partial\mathbb{U}$" might characterize the strongly asymptotically Toeplitz composition operators. The extreme case (inner functions) provides further support (Theorem 3.3), but, somewhat to our surprise, the conjecture turns out to be false in general (Theorem 3.4), and we have no simple alternative to offer. The last section initiates, for composition operators, the study of "conjugate asymptotic toeplitzness," a concept that is interesting only for the strong operator topology. Our main result here (Theorem 4.2) is that, except for trivial cases, C_φ^* is strongly asymptotically Toeplitz whenever φ fixes the origin. So far little of substance is known about what happens when the origin is not fixed.

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1 Uniform Asymptotic Toeplitzness

To state the definition fully: a bounded operator T on H^2 is *uniformly asymptotically Toeplitz* provided there is a (necessarily bounded) operator A on H^2 such that

$$\lim_{n\to\infty} \|S^{*n}TS^n - A\| = 0. \tag{1}$$

This class of operators is easily seen to be a subspace—in fact one that is closed both in norm and under adjoints—of the space $\mathcal{L}(H^2)$ of all bounded linear operators on H^2. It contains both the Toeplitz operators, which satisfy (1) with $A = T$, and the compact ones, which satisfy (1) with $A = 0$. Hence any operator of the form "Toeplitz plus compact" is uniformly asymptotically Toeplitz. Feintuch assures us that there are no others:

Theorem F [6, Theorem 2.4]. *An operator in $\mathcal{L}(H^2)$ is uniformly asymptotically Toeplitz if and only if it has the form Toeplitz + compact.*

Since [6] may not be readily accessible to all readers, we outline the proof at the end of this section. Right now let's address the main point of this section: Which *composition operators* are uniformly asymptotically Toeplitz?

First a more basic question: *Which composition operators are Toeplitz?* This one is easy: *only the identity operator.* Indeed, the matrix of C_φ has as its columns the Maclaurin coefficients of the successive powers $1 = \varphi^0, \varphi, \varphi^2, \ldots$ of φ. So if the diagonals of this matrix are constant, then upon looking first at the main diagonal and then at the successive subdiagonals we see that $\hat{\varphi}(1) = 1$ and $\hat{\varphi}(n) = 0$ for $n > 1$, where "$\widehat{\ }$" denotes "Maclaurin coefficient." A look at the first superdiagonal shows that

$$\hat{\varphi}(0) = \widehat{\varphi^2}(1) = 2\hat{\varphi}(0)\hat{\varphi}(1) = 2\hat{\varphi}(0),$$

hence $\hat{\varphi}(0) = 0$. Thus $\varphi(z) \equiv z$, i.e., C_φ is the identity operator on H^2.

Consequently, the only composition operators that are obviously uniformly asymptotically Toeplitz are the compact ones (of which there are many, see [3, 15, 16] for example) and the identity. Thanks to Feintuch's Theorem and some standard tools of composition operator technology, we can show that there are no others.

THEOREM 1.1 *A composition operator on H^2 is uniformly asymptotically Toeplitz if and only if it is either compact or the identity.*

Proof Given the discussion above, we need only show that if C_φ is neither compact nor the identity, then its difference with any Toeplitz operator fails to be compact.

For this we use the fact that Toeplitz operators on H^2 are those of the form

$$T_b f = P(bf) \qquad (f \in H^2)$$

where b is a bounded measurable function (called the "symbol" of the operator) on the unit circle $\partial\mathbb{U}$, and P is the orthogonal projection of $L^2 = L^2(\partial\mathbb{U})$ onto H^2, where now we view H^2 as the closed subspace of L^2 consisting of functions whose Fourier coefficients of negative index are all zero [4, Chapter 7, page 177]. Fix a composition operator C_φ, neither the identity nor compact, and a Toeplitz operator T_b, and set $\Delta = T_b - C_\varphi$. We claim Δ is not compact.

We may as well assume b is not a.e. zero on $\partial\mathbb{U}$, since otherwise T_b is the zero-operator and then $\Delta = -C_\varphi$, which we are assuming noncompact. It is enough to show that the adjoint operator $\Delta^* = T_b^* - C_\varphi^*$ is not compact; for this employ a convenient collection of unit vectors: $k_a = K_a/\|K_a\|$, where K_a is the *reproducing kernel* for the point for $a \in \mathbb{U}$:

$$K_a(z) = \frac{1}{1 - \overline{a}z} \qquad (z \in \mathbb{U}).$$

The nomenclature comes from the"reproducing property"

$$\langle f, K_a \rangle = f(a) \qquad (f \in H^2),$$

which in turn leads to the norm computation

$$\|K_a\|^2 = \langle K_a, K_a \rangle = K_a(a) = \frac{1}{1 - |a|^2} \qquad (a \in \mathbb{U}).$$

From this it follows easily that the "normalized reproducing kernels" k_a converge weakly to zero as $|a| \to 1-$. Another easy consequence of the reproducing property is the *Adjoint Formula:*

$$C_\varphi^* K_a = K_{\varphi(a)} \qquad (a \in \mathbb{U}). \tag{2}$$

This is all standard fare that can be found, for example, in [16, pp. 43–44].

To show Δ^* is not compact it suffices to show, thanks to the weak convergence of k_a to zero, that $\{\|\Delta^* k_a\|\}$ does not converge to zero as $|a| \to 1-$.

We'll prove something even better: $\{\langle \Delta^* k_a,\, k_a\rangle\}$ does not converge to zero.

Using (2) and the fact that $T_b^* = T_{\overline{b}}$ (see [4, Proposition 7.4, page 178], for example) we have

$$\begin{aligned}\langle \Delta^* k_a,\, k_a\rangle &= \langle T_{\overline{b}}\, k_a,\, k_a\rangle - \langle C_\varphi^*\, k_a,\, k_a\rangle \\ &= \langle P(\overline{b}k_a),\, k_a\rangle - (1-|a|^2)\langle K_{\varphi(a)},\, K_a\rangle \\ &= \langle \overline{b}k_a,\, Pk_a\rangle - (1-|a|^2)K_{\varphi(a)}(a) \\ &= \langle \overline{b}k_a,\, k_a\rangle - \frac{1-|a|^2}{1-\overline{\varphi(a)}a}\ ,\end{aligned}$$

where in the final two lines we used, successively, the self-adjointness of the projection P and the fact that k_a is in H^2, the range of this projection. Now

$$\langle \overline{b}k_a,\, k_a\rangle = \int_{\partial\mathbb{U}} \overline{b}|k_a|^2\, dm = \int_{\partial\mathbb{U}} \overline{b(\zeta)}\frac{1-|a|^2}{|1-\overline{a}\zeta|^2}\, dm(\zeta)\,,$$

where dm denotes Lebesgue measure on $\partial\mathbb{U}$, normalized to have total mass one. The final integral here is just $\mathcal{P}\left[\,\overline{b}\,\right](a)$, the Poisson integral of the function $\overline{b}$, evaluated at the point a. Putting everything together, we have

$$\langle \Delta^* k_a,\, k_a\rangle = \mathcal{P}\left[\,\overline{b}\,\right](a) - \frac{1-|a|^2}{1-\overline{\varphi(a)}a} \qquad (a \in \mathbb{U}). \tag{3}$$

Since C_φ is not the identity operator on H^2, the map φ is not the identity on $\mathbb{U}$. By the "boundary uniqueness" property of bounded holomorphic functions ([5, Theorem 2.2, page 17], [14, Theorem 17.18, page 345]) the set

$$E = \{\zeta \in \mathbb{U} : \varphi(\zeta) \neq \zeta\}$$

therefore has full measure: $m(E) = 1$, where $\varphi(\zeta)$ denotes the radial limit of φ at ζ (which exists at m-a.e. point of $\partial\mathbb{U}$, see [5, Theorem 1.3, page 6] or [14, Theorem 17.11, page 340]). Recall that we are assuming b is not a.e. zero on $\partial\mathbb{U}$, i.e., that the set

$$F = \{\zeta \in \partial\mathbb{U} : b(\zeta) \neq 0\}$$

has positive measure. Thus $E \cap F$ has positive measure, and in particular is not empty. Fix a point $\zeta \in E \cap F$. We have, as $r \to 1-$,

$$\mathcal{P}[\bar{b}](r\zeta) \to \bar{b}(\zeta) \neq 0 \quad \text{and} \quad \frac{1 - r^2}{1 - \overline{\varphi(r\zeta)}r\zeta} \to \frac{0}{1 - \overline{\varphi(\zeta)}\zeta} = 0,$$

(the denominator of the last fraction is not zero because $\varphi(\zeta) \neq \zeta$). This combines with (3) to show that

$$\lim_{r \to 1-} \langle \Delta^* k_{r\zeta},\, k_{r\zeta} \rangle = \bar{b}(\zeta) \neq 0,$$

thus establishing that Δ^*, and therefore Δ, is not compact. □

Feintuch's proof of Theorem F. We have already noted that every compact perturbation of a Toeplitz operator is uniformly asymptotically Toeplitz. For the converse, suppose that T is uniformly asymptotically Toeplitz, i.e., that (1) holds for some $A \in \mathcal{L}(H^2)$. Note that A is a Toeplitz operator: $S^*AS = A$, hence

$$S^{*n}TS^n - A = S^{*n}(T - A)S^n$$

for each n. Let P_n denote the orthogonal projection taking H^2 onto the closed linear span of the monomials $\{z^k : k \geq n\}$, and observe that $\|S^{*n}f\| = \|P_n f\|$ and that $P_n = S^n S^{*n}$. Putting it all together:

$$\|P_n(T - A)P_n\| = \|S^{*n}(T - A)S^n S^{*n}\| \leq \|S^{*n}(T - A)S^n\|$$

hence

$$\lim_{n \to \infty} \|P_n(T - A)P_n\| = 0 \tag{4}$$

Now $P_n = I - Q_n$, where Q_n is the orthogonal projection onto the linear span of $\{z^k, 0 \leq k < n\}$. Thus

$$P_n(T - A)P_n = (I - Q_n)(T - A)(I - Q_n) = T - A + F_n$$

where F_n is a finite-rank operator. Viewed this way, (4) tells us that $T - A$ is a norm-limit of finite rank operators, hence is compact. Thus T has the desired form "Toeplitz plus compact." □

2 Weak Asymptotic Toeplitzness

Recall that we define $T \in \mathcal{L}(H^2)$ to be *weakly asymptotically Toeplitz* (henceforth "WAT") whenever the sequence of operators $\{S^{*n}TS^n\}$ converges in the weak operator topology of H^2, i.e., whenever there exists $A \in \mathcal{L}(H^2)$ such that

$$\lim_{n\to\infty} \langle S^{*n}TS^n f,\, g\rangle = \langle Af,\, g\rangle \qquad \forall\ f, g \in H^2. \tag{5}$$

Remarks. (a) In (5) it's clearly enough to consider f and g to be monomials, from which we see that an operator is weakly asymptotically Toeplitz if and only if its matrix (with respect to the monomial basis for H^2) has convergent diagonals. In summary: "constant diagonals" means "Toeplitz," while "convergent diagonals" means "weakly asymptotically Toeplitz."

(b) As in the previous section, the operator A satisfies the equation $S^*AS = A$, i.e., it is Toeplitz. Its symbol is called the *asymptotic symbol* of T. For example, compact operators have asymptotic symbol $\equiv 0$, so according to Theorem 1.1, for uniformly asymptotically Toeplitz composition operators the only possible asymptotic symbols are the constants 1 (for the identity operator) and 0 (for compacts). This "zero-one dichotomy" for asymptotic symbols will pervade the rest of our work.

(c) *If φ is a nontrivial rotation, then C_φ is not weakly asymptotically Toeplitz.*

Proof We have $\varphi(z) \equiv \omega z$ for some $\omega \in \partial\mathbb{U}\backslash\{1\}$. Thus

$$\langle S^{*n}C_\varphi S^n f,\, g\rangle = \omega^n \langle f\circ\varphi,\, g\rangle \qquad (f, g \in H^2), \tag{6}$$

and since the scalar sequence $\{\omega^n\}$ does not converge, neither does the operator sequence $\{S^{*n}C_\varphi S^n\}$ (weakly). Thus C_φ is not WAT. $\square$

We show next that the notions of weak and uniform asymptotic toeplitzness pick out dramatically different classes of composition operators.

Proposition 2.1 *Suppose φ, neither the identity nor a rotation, fixes the origin. Then C_φ is weakly asymptotically Toeplitz with asymptotic symbol $\equiv 0$.*

Proof Since φ fixes the origin and is neither the identity nor a rotation, the function ψ defined on $\mathbb{U}$ by $\psi(z) = \varphi(z)/z$ is a nonconstant holomorphic selfmap of $\mathbb{U}$. In particular, $|\varphi| < 1$ at every point of $\mathbb{U}$. Now for $f, g \in H^2$,

$$\langle S^{*n}C_\varphi S^n f,\, g\rangle = \langle \varphi^n \cdot (f\circ\varphi),\, z^n g\rangle = \langle \psi^n \cdot (f\circ\varphi),\, g\rangle = \int_{\partial\mathbb{U}} \psi^n \cdot (f\circ\varphi)\,\overline{g}\,dm\,. \tag{7}$$

The sequence $\{\psi^n \cdot (f \circ \varphi)\}$ is bounded in H^2, and convergent pointwise to zero, hence it converges weakly to zero in H^2, i.e., the left hand side of (7) converges to zero for each pair of functions f and g in H^2. Thus $C_\varphi \in$ WAT with asymptotic symbol $\equiv 0$. □

WAT Conjecture. *If φ is neither a rotation nor the identity map, then C_φ is weakly asymptotically Toeplitz with asymptotic symbol $\equiv 0$.*

We already know the conjecture holds (trivially) for maps φ that induce compact composition operators, and by Proposition 2.1 it holds also for maps that fix the origin. At the end of this section, and in the next one, we'll show that it holds for many maps φ that neither fix the origin nor induce compact composition operators. But first we prove a weaker version of the conjecture. To state this result efficiently, let's say that $T \in \mathcal{L}(H^2)$ is "mean weakly asymptotically Toeplitz" (henceforth: "MWAT") if the *arithmentic means* of the sequence $\{S^{*n}TS^n\}$ converge in the weak operator topology of H^2. One checks easily that when this happens the limit operator is Toeplitz, so each MWAT operator has an asymptotic symbol.

Theorem 2.2 *Every composition operator on H^2 is MWAT. Except for the identity, each has asymptotic symbol $\equiv 0$.*

Proof Note first that, by (6), non-trivial rotations induce MWAT composition operators with asymptotic symbol zero.

Suppose φ is neither the identity nor a rotation. For N a non-negative integer let

$$\Gamma_N = \frac{1}{N+1}\sum_{n=0}^{N} S^{*n}C_\varphi S^n\,.$$

As in the previous proof, it's enough to check weak operator convergence by using only monomials in the inner product, so our goal is to show that for each pair α, β of non-negative integers,

$$\langle \Gamma z^\alpha,\, z^\beta\rangle = \left\langle \frac{1}{N+1}\sum_{n=0}^{N}(S^{*n}C_\varphi S^n)z^\alpha, z^\beta \right\rangle \to 0 \qquad (n\to\infty). \tag{8}$$

To this end, fix non-negative integers α, β, and n, and observe that

$$\langle S^{*n}C_\varphi S^n z^\alpha,\, z^\beta\rangle = \int_{\partial\mathbb{U}} \varphi^{n+\alpha}(\zeta)\,\overline{\zeta^{n+\beta}}\,dm(\zeta) = \int_{\partial\mathbb{U}} \psi^{n+\alpha}\,d\mu \tag{9}$$

where $\psi : \partial\mathbb{U} \to \overline{\mathbb{U}}$ is defined by

$$\psi(\zeta) := \overline{\zeta}\varphi(\zeta) = \varphi(\zeta)/\zeta \qquad (\zeta \in \partial\mathbb{U}), \tag{10}$$

(cf. the proof of Proposition 2.1) and μ is the measure defined on the Borel sets of $\partial\mathbb{U}$ by

$$d\mu(\zeta) := \zeta^{\alpha-\beta}\, dm(\zeta)\,. \tag{11}$$

Let $E := \varphi^{-1}(\partial\mathbb{U}) \cap \partial\mathbb{U}$, i.e.,

$$E = \{\zeta \in \partial\mathbb{U} : |\varphi(\zeta)| = 1\} = \{\zeta \in \partial\mathbb{U} : |\psi(\zeta)| = 1\}.$$

Since $\psi^n \to 0$ pointwise on $\partial\mathbb{U}\backslash E$ we have, thanks to the Bounded Convergence Theorem,

$$\lim_{n\to\infty} \int_{\partial\mathbb{U}\backslash E} \psi^n\, d\mu = 0,$$

so by (9), to prove (8) we need only show that

$$\lim_{n\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} \int_E \psi^n\, d\mu = 0. \tag{12}$$

Let ν denote the restriction of the measure μ to E. Since $\psi(E) \subset \partial\mathbb{U}$, the pullback measure $\nu\psi^{-1}$ is a Borel measure on $\partial\mathbb{U}$, so the change of variable formula yields for each non-negative integer n:

$$\int_E \psi^n\, d\mu = \int_{\partial\mathbb{U}} \psi^n\, d\nu = \int_{\partial\mathbb{U}} \zeta^n\, d\nu\psi^{-1}(\zeta) = \widehat{\nu\psi^{-1}}(-n), \tag{13}$$

where "$\widehat{\ }$" now denotes "Fourier coefficient."

The crucial observation about the measure $\nu\psi^{-1}$ is that *it has no mass point*. Indeed, since φ is a bounded analytic function on $\mathbb{U}$ that is neither the identity nor a rotation, the "boundary uniqueness theorem" that anchored the proof of Theorem 1.1 insures that for each point $\omega \in \partial\mathbb{U}$ the set

$$\psi^{-1}(\{\omega\}) = \{\zeta \in \partial\mathbb{U} : \psi(\zeta) = \omega\} = \{\zeta \in \partial\mathbb{U} : \varphi(\zeta) = \omega\zeta\}$$

has Lebesgue measure zero. Since $\nu << m$ we see that $\nu\psi^{-1}(\{\omega\}) = 0$, as claimed.

Thus by Weiner's Mass Point Theorem (see [8, page 42], for example)

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\widehat{\nu\psi^{-1}}(n)|^2 = 0,$$

from which we require only the weaker statement

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\widehat{\nu\psi^{-1}}(n)| = 0,$$

which, by (13), implies (12). □

Note that if we can prove $\nu\psi^{-1}$ to be *absolutely continuous* with respect to Lebesgue measure, then, by the Riemann-Lebesgue Lemma, we will have established the truth of the WAT Conjecture. Here is a special case in which we can do this.

PROPOSITION 2.3 *Suppose φ is neither the identity nor a rotation. If $|\varphi| = 1$ on an open subset V of $\partial\mathbb{U}$, and $|\varphi| < 1$ a.e. on $\partial\mathbb{U}\backslash V$, then C_φ is* WAT *with essential symbol $\equiv 0$.*

Proof Fix non-negative integers α and β. By analogy with the proof of Theorem 2.2 it will be enough to prove that

$$\lim_{n\to\infty} \langle S^{*n} C_\varphi S^n z^\alpha,\, z^\beta\rangle = 0.$$

By (9) we have:

$$\lim_{n\to\infty} \langle S^{*n} C_\varphi S^n z^\alpha,\, z^\beta\rangle = \int_V + \int_{\partial\mathbb{U}\backslash V} \psi^{n+\alpha}\, d\mu,$$

where ψ is given by (10) and μ by (11).

Since $|\psi| = |\varphi| < 1$ a.e. on $\partial\mathbb{U}\backslash V$, the Lebesgue bounded convergence theorem sends the second integral to zero as $n \to \infty$. Thus we need only show that

$$\lim_{n\to\infty} \int_V \psi^{n+\alpha}\, d\mu = 0. \tag{14}$$

For this it is more convenient to work on the real line. Let V^* be the relatively open subset of $[0, 2\pi)$ that is sent to V by the map $t \to e^{it}$, and let

$$d\mu^*(t) = e^{i(\alpha-\beta)t} \frac{dt}{2\pi} \,,$$

the measure that corresponds under this map to $d\mu$.

Because its values have modulus one on the open subset V of $\partial\mathbb{U}$, the map φ extends by reflection analytically over V. Thus for $t \in V^*$ we have

$$\varphi(e^{it}) = e^{i\gamma(t)} \quad \text{and} \quad \psi(e^{it}) = e^{i\gamma(t)-t} \equiv e^{i\delta(t)},$$

with γ, and hence δ, real-analytic on V^*. The change-of-variable formula now yields

$$\int_V \psi^{n+\alpha}\, d\mu = \int_{V^*} e^{in\delta(t)}\, d\mu^*(t) = \int_{\delta(V^*)} e^{ins}\, d\mu^*\delta^{-1}(s)\,.$$

Thus to prove (14) we need only show that on $\delta(V^*)$ the measure $\mu^*\delta^{-1}$ is absolutely continuous with respect to Lebesgue measure.

For this, note that V^* is an at most countable disjoint union of intervals, hence $\delta(V^*)$ is the union of the δ-images of these intervals, each of which is also an interval. Each such image interval is exhausted by the images of the compact subintervals of the pre-image interval, so it is enough to prove that $\mu^*\delta^{-1}$ is absolutely continuous with respect to Lebesgue measure on each of the subintervals $\delta(I)$, where I is a compact subinterval of V^*. Since δ is real-analytic on a neighborhood of I, its derivative has at most finitely many sign changes on I, hence I splits into a finite collection of contiguous subintervals, on each of which δ is strictly monotonic. On each of these subintervals the desired absolute continuity of $\mu^*\delta^{-1}$ follows from the change-of-variable formula of elementary Calculus. □

Proposition 2.3 is particularly appealing when applied to inner functions. Recall that the *singular set* of an inner function is the closed subset of $\partial\mathbb{U}$ consisting of points over which the function has no analytic continuation. It is the union of the limit points of zeros of the Blaschke factor and the closed support of the measure associated with the singular factor.

Corollary 2.4 *Suppose φ, neither the identity nor a rotation, is an inner function with singular set of measure zero. Then C_φ is weakly asymptotically Toeplitz with asymptotic symbol $\equiv 0$.*

3 Strong Asymptotic Toeplitzness

To say that an operator $T \in \mathcal{L}(H^2)$ is *strongly asymptotically Toeplitz* (henceforth: "SAT") means that the operator sequence $\{S^{*n}TS^n\}$ converges strongly on H^2, i.e. there exists a (necessarily Toeplitz) operator A on H^2 such that

$$\lim_{n\to\infty} \|S^{*n}TS^n f - Af\| = 0$$

for every $f \in H^2$.

PROPOSITION 3.1 *Suppose $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$. Then C_φ is strongly asymptotically Toeplitz with asymptotic symbol $\equiv 0$.*

Proof Fix $f \in H^2$ and observe that, because $\|S^*\| = 1$,

$$\|S^{*n}C_\varphi S^n f\|^2 \leq \|C_\varphi S^n f\|^2 = \|\varphi^n f\circ\varphi\|^2 = \int_{\partial\mathbb{U}} |\varphi|^{2n}|f\circ\varphi|^2\,dm.$$

The hypothesis on φ guarantees that $|\varphi|^n \to 0$ a.e. on $\partial\mathbb{U}$ so, because $f\circ\varphi \in H^2$, we see that $\|S^{*n}C_\varphi S^n f\|^2 \to 0$ by the Lebesgue Dominated Convergence Theorem. Thus C_φ is SAT and its asymptotic symbol is $\equiv 0$. $\square$

Here is a partial converse to Proposition 3.1 which shows that many composition operators fail to be strongly asymptotically Toeplitz.

PROPOSITION 3.2 *Suppose φ, not the identity map, fixes the origin. If C_φ is strongly asymptotically Toeplitz then $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$.*

Proof Suppose φ is asymptotically Toeplitz. Since φ is not the identity map, Proposition 2.1 insures that the essential symbol of C_φ is $\equiv 0$; in particular

$$\lim_{n\to\infty} \|S^{*n}C_\varphi S^n 1\| \to 0. \tag{15}$$

As in the proof of Proposition 2.1 write $\varphi(z) = z\psi(z)$ where ψ is holomorphic in $\mathbb{U}$. Our goal is to show that

$$E = \{\zeta \in \partial\mathbb{U} : |\varphi(\zeta)| = 1\}$$

has measure zero. For each non-negative integer n:

$$\begin{aligned}\|S^{*n}C_\varphi S^n 1\|^2 &= \int_{\partial\mathbb{U}} |S^{*n}\varphi^n|^2\,dm = \int_{\partial\mathbb{U}} |S^{*n}z^n\psi^n|^2\,dm\\ &= \int_{\partial\mathbb{U}} |\psi|^{2n}\,dm \geq m(E)\end{aligned}$$

which, by (15) shows that $m(E) = 0$. $\square$

Propositions 3.1 and 3.2 lead one to suspect that the hypothesis "$|\varphi| = 1$ a.e. on a subset of $\partial\mathbb{U}$ having positive measure" might *always* disqualify C_φ from being strongly asymptotically Toeplitz. The next two results show that this embryonic conjecture is in part true, but in larger part false.

THEOREM 3.3 *If φ is an inner function other than the identity map, then C_φ is not strongly asymptotically Toeplitz.*

Proof The case $\varphi(0) = 0$ has already been handled by Proposition 3.2. So we assume from now on that $\varphi(0) = a \neq 0$.

As in the proof of Proposition 3.2 it will suffice to show—this time thanks to Theorem 2.2—that the norms of the vectors $S^{*n}C_\varphi S^n 1 = S^{*n}\varphi^n$ are bounded away from zero, i.e.,

$$\inf_n \sum_{k=n}^{\infty} |\widehat{\varphi^n}(k)|^2 > 0. \tag{16}$$

It turns out that we need only prove this for conformal automorphisms of the unit disc. Indeed, let ψ be an automorphism of $\mathbb{U}$ with $\psi(0) = a$. Then $\omega = \psi^{-1} \circ \varphi$ is an inner function that fixes the origin, so $\varphi = \psi \circ \omega$, hence also $\varphi^n = \psi^n \circ \omega$ for each non-negative integer n. A corollary of Littlewood's Subordination Theorem [11, Theorem 215, page 168] now asserts that

$$\sum_{k=0}^{n} |\widehat{\varphi^n}(k)|^2 \leq \sum_{k=0}^{n} |\widehat{\psi^n}(k)|^2 \qquad (n = 0, 1, 2, \ldots).$$

This, and the fact that $\|\varphi^n\| = \|\psi^n\| = 1$ (both φ^n and ψ^n are inner) shows that

$$\sum_{k=n}^{\infty} |\widehat{\psi^n}(k)|^2 \leq \sum_{k=n}^{\infty} |\widehat{\varphi^n}(k)|^2 \qquad (n = 0, 1, 2, \ldots) \tag{17}$$

(see also [13, page 254] and [12, pp. 368–369]).

Thus we need only show that the sum on the left-hand side of (17) is bounded away from zero. For this, observe first that, since ψ is a conformal automorphism of $\mathbb{U}$, it is analytic across the unit circle, so

$$1 = \text{normalized arclength of } \psi(\partial\mathbb{U}) = \int_{\partial\mathbb{U}} |\psi'| \, dm,$$

hence by the Cauchy-Schwarz inequality, $1 < \int_{\partial\mathbb{U}} |\psi'|^2\, dm$, the strict inequality arising from the fact that $|\psi'| \neq 1$ a.e. on $\partial\mathbb{U}$ (from the assumption that $\psi(0) \neq 0$). To say this another way: *There exists a constant $d > 0$ such that*

$$\int_{\partial\mathbb{U}} |\psi'|^2\, dm = 1 + d. \tag{18}$$

Now fix a positive integer n and temporarily set $f = \psi^n$. Then on $\partial\mathbb{U}$:

$$|f'| = |n\psi^{n-1}\psi'| = n|\psi'|,$$

which, along with (18) yields

$$\sum_{k=1}^{\infty} k^2\, |\widehat{\psi^n}(k)|^2 = \int_{\partial\mathbb{U}} |f'|^2\, dm = n^2 \int_{\partial\mathbb{U}} |\psi'|^2 dm = n^2(1+d). \tag{19}$$

Finally, observe that upon estimating $\int_{\partial\mathbb{U}} |f''|^2\, dm$ by using the analyticity of ψ across $\partial\mathbb{U}$ to bound $|\psi'|$ and $|\psi''|$ on $\partial\mathbb{U}$, we obtain a constant A, *which does not depend on n,* such that

$$\sum_{k=1}^{\infty} k^4\, |\widehat{\psi^n}(k)|^2 \leq A\, n^4. \tag{20}$$

Fix a positive integer α with

$$\alpha^2 > \frac{2A}{d}\ . \tag{21}$$

We will show that for each n:

$$\sum_{k=n}^{\alpha n} |\widehat{\psi^n}(k)|^2 > \frac{d}{2\alpha^2}, \tag{22}$$

thus establishing the lower bound (16) with ψ in place of φ, and so, by inequality (17), finishing the proof.

For this, let σ_n denote the sum on the left-hand side of (22), and observe that

$$(\alpha n)^2 \sigma_n \geq \sum_{k=n}^{\alpha n} k^2\, |\widehat{\psi^n}(k)|^2$$

$$= \left\{ \sum_{k=1}^{\infty} - \sum_{k=1}^{n-1} - \sum_{k=\alpha n+1}^{\infty} \right\} k^2 \, |\widehat{\psi^n}(k)|^2$$

$$\equiv \quad \Sigma_1 \quad - \quad \Sigma_2 \quad - \quad \Sigma_3.$$

Now $\Sigma_1 = n^2(1+d)$ by (19), while clearly

$$\Sigma_2 \leq n^2 \sum_{k=1}^{n-1} |\widehat{\psi^n}(k)|^2 \leq n^2 \sum_{k=1}^{\infty} |\widehat{\psi^n}(k)|^2 = n^2 \|\psi^n\| = n^2.$$

Finally, by (20),

$$\Sigma_3 \leq \sum_{k=\alpha n+1}^{\infty} \frac{k^4}{(\alpha n)^2} |\widehat{\psi^n}(k)|^2 \leq \frac{A}{\alpha^2} n^2.$$

Putting this all together and using (21):

$$(\alpha n)^2 \sigma_n \geq (d - \frac{A}{\alpha^2}) n^2 > \frac{d}{2} n^2$$

which establishes (22) and finishes the proof. □

The next result shows that for maps φ that do not fix the origin, the condition of having radial limits of modulus one on a set of positive measure—even on a nontrivial arc—does not rule out strong asymptotic toeplitzness. For convenience we use the notation $\|f\|_\infty$ to denote the supremum of $|f(z)|$ over $z \in \mathbb{U}$.

THEOREM 3.4 *Suppose φ extends continuously to a nontrivial closed arc J of $\partial\mathbb{U}$ which it maps into $\partial\mathbb{U}$. Suppose further that $|\varphi| < 1$ a.e. on $\partial\mathbb{U}\backslash J$, and that $\|\varphi'\|_\infty < 1$. Then C_φ is strongly asymptotically Toeplitz.*

Remarks (a) The condition $\|\varphi'\|_\infty < 1$ insures that φ cannot fix the origin (if φ did fix the origin, we would obtain, upon representing $\varphi(z)$ as the integral on the ray from 0 to z of φ', the inequality $|\varphi(z)| \leq \|\varphi'\|_\infty$, hence the contradiction $\|\varphi\|_\infty \leq \|\varphi'\|_\infty < 1$), so there is no obvious conflict between Theorem 3.4 and Proposition 3.2.

(b) Maps φ that satisfy the hypotheses of Theorem 3.4 are easy to construct. For example, let Ω be a C^2 Jordan domain in the open upper half-plane Π_+, whose boundary intersects the real line in an interval I. Let ψ be a Riemann map taking $\mathbb{U}$ onto Ω, so ψ extends C^1 to the unit circle[1], takes some closed arc J of $\partial\mathbb{U}$ onto I, and takes the rest of $\partial\mathbb{U}$ into $\mathbb{U}$. Let $\tau(w) = (1+iw)/(1-iw)$, a Möbius map taking Π_+ onto $\mathbb{U}$ and, for $a > 0$ to be chosen shortly, set $\varphi(z) = \tau(a\psi(z))$. Then φ maps $\mathbb{U}$ univalently onto $\tau(a\Omega)$, and so takes the arc J onto a nontrivial arc of $\partial\mathbb{U}$ while sending the rest of the closed unit disc into $\mathbb{U}$. Moreover, $|\tau'| < 2$ on Π_+, so by the Chain Rule, for a sufficiently small:

$$\|\varphi'\|_\infty \leq 2a\|\psi'\|_\infty < 1,$$

hence φ satisfies the hypotheses of Theorem 3.4.

Proof of Theorem 3.4 It is enough to prove that for each non-negative integer α,

$$\lim_{n\to\infty} \|S^{*n}C_\varphi S^n z^\alpha\| = 0.$$

By Proposition 2.3 we know that $C_\varphi \in$ WAT, so for each non-negative integer j,

$$\widehat{\varphi^{n+\alpha}}(n+j) = \langle S^{*n}C_\varphi S^n z^\alpha,\, z^j\rangle \to 0$$

as $n \to \infty$. Thus

$$\begin{aligned}
\|S^{*n}C_\varphi S^n z^\alpha\|^2 &= \sum_{k=n}^{\infty} |\widehat{\varphi^{n+\alpha}}(n+k)|^2 \\
&= \left\{ \sum_{k=n}^{n+\alpha-1} + \sum_{k=n+\alpha}^{\infty} \right\} |\widehat{\varphi^{n+\alpha}}(n+k)|^2 \\
&= \mathrm{o}(1) + \sum_{k=n+\alpha}^{\infty} |\widehat{\varphi^{n+\alpha}}(k)|^2
\end{aligned}$$

as $n \to \infty$. It therefore suffices to show that

$$\lim_{n\to 0} \sum_{k=n}^{\infty} |\widehat{\varphi^n}(k)|^2 = 0 \tag{23}$$

(i.e., that $\lim_n \|S^{*n}C_\varphi S^n 1\| = 0$).

[1]See, e.g., [9]. If Ω has C^∞ boundary then the extension is C^∞ on $\partial\mathbb{U}$; this is the classical theorem of Painlevé—for a nice exposition see [2].

Now let $\varepsilon > 0$ be given. Choose a closed arc J_1 of $\partial\mathbb{U}$ that lies in the interior of the (closed) arc J so that $m(J\backslash J_1) < \varepsilon/4$. Set $J_2 = J\backslash J_1$ and $J_3 = \partial\mathbb{U}\backslash J$. Thus $\partial\mathbb{U}$ is the disjoint union of J_1, J_2, and J_3, and for each n there is a corresponding decomposition of φ^n into three pieces:

$$\Phi_{n,j} := \varphi^n I_j \qquad (j = 1, 2, 3),$$

where I_j is the indicator (or "characteristic") function of J_j.

We will consider in turn each of the sums

$$\Sigma_{j,n} := \sum_{k=n}^{\infty} |\widehat{\Phi_{j,n}}(k)|^2 \qquad (j = 1, 2, 3).$$

Recall that $m(J_2) < \varepsilon/4$, hence for all n:

$$\Sigma_{2,n} \leq \sum_{k=0}^{\infty} |\widehat{\Phi_{2,n}}(k)|^2 = \int_{J_2} |\varphi|^{2n}\, dm \leq \varepsilon/4. \tag{24}$$

Since $|\varphi| < 1$ on J_3 we have

$$\Sigma_{3,n} \leq \sum_{k=0}^{\infty} |\widehat{\Phi_{3,n}}(k)|^2 = \int_{J_3} |\varphi|^{2n}\, dm \to 0$$

as $n \to \infty$, hence there exists $N_3 > 0$ such that

$$n \geq N_3 \Rightarrow \Sigma_{3,n} < \varepsilon/4. \tag{25}$$

The interesting sum is $\Sigma_{1,n}$. Since φ maps J to a nontrivial arc of the unit circle, it extends by reflection to a function analytic in a neighborhood of J, hence for $e^{it} \in J$

$$\varphi(e^{it}) = e^{i\theta(t)}$$

where θ is a real-analytic on the interior of J—or rather on the interior of the interval of $[0, 2\pi]$ that corresponds to J, which we henceforth identify with J. By the chain rule,

$$|\theta'(t)| = |\varphi'(e^{it})| \leq \|\varphi'\|_\infty < 1$$

whenever $t \in J$. Moreover, θ'', being also real-analytic in J, has only finitely many sign changes—say ν of them—in the closed subarc J_1, hence J_1 decomposes into ν subarcs $K_1, \dots, K_\nu$ on each of which θ' is monotonic. Now for n and k non-negative integers,

$$\widehat{\Phi_{1,n}}(k) = \int_{J_1} e^{i\beta(t)}\, dt,$$

where

$$\beta(t) := n\theta(t) - kt,$$

also real-analytic with derivative monotonic on each of the intervals K_j ($j = 1, \dots, \nu$). Thus van der Corput's Lemma (see, e.g., [17, Proposition 2(ii), page 332]) applies on each of the intervals K_j, and yields

$$\left| \int_{K_j} e^{i\beta(t)}\, dt \right| \le \frac{3}{\min_{K_j} |\beta'|} \qquad (j = 1, \dots \nu). \tag{26}$$

Now for $t \in J$ and $k \ge n$:

$$|\beta'(t)| = |n\theta'(t) - k| \ge k - n|\theta'(t)| \ge k - n\|\varphi'\|_\infty \ge k(1 - \|\varphi'\|_\infty),$$

so by (26)

$$\left| \int_{K_j} e^{i\beta(t)}\, dt \right| \le \frac{C}{k} \qquad (j = 1, \dots \nu),$$

with $C = 3/(1 - \|\varphi'\|_\infty)$. Thus

$$|\widehat{\Phi_{1,n}}(k)| \le \frac{C\nu}{k} \qquad (k \ge n)$$

from which it follows that

$$\Sigma_{1,n} \le (C\nu)^2 \sum_{k=n}^{\infty} \frac{1}{k^2} \le \frac{(C\nu)^2}{n-1}.$$

So upon choosing N_1 sufficiently large we insure that $\Sigma_{1,n} < \varepsilon/2$ whenever $n \ge N_1$. From this, (24), and (25) we see that if $n \ge \max(N_1, N_3)$ then

$$\sum_{k=n}^{\infty} |\widehat{\varphi^n}(k)|^2 = \Sigma_{1,n} + \Sigma_{2,n} + \Sigma_{3,n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon,$$

and we are done. □

4 Adjoint Asymptotic Toeplitzness

Let us recall the flavors of asymptotic toeplitzness discussed so far:

$$\text{UAT} \;\Rightarrow\; \text{SAT} \;\Rightarrow\; \text{WAT} \;\Rightarrow\; \text{MWAT}.$$

Here is a complement to Proposition 3.1 that provides a simple sufficient condition for the adjoint of a composition operator to be strongly asymptotically Toeplitz.

PROPOSITION 4.1 *If $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$ then C_φ^* is strongly asymptotically Toeplitz.*

Proof Fix f and n. Then choose g a unit vector in H^2 such that $\|S^{*n}C_\varphi^*S^n f\| = \langle S^{*n}C_\varphi^*S^n f,\, g\rangle$. Thus

$$\begin{aligned}
\|S^{*n}C_\varphi^*S^n f\| &= \langle S^n f,\, C_\varphi S^n g\rangle = \langle z^n f,\, \varphi^n\, g\circ\varphi\rangle \\
&\le \int_{\partial\mathbb{U}} |f|\,|g\circ\varphi||\varphi|^n\,dm \\
&\le \left\{\int_{\partial\mathbb{U}} |f|^2\,|\varphi|^{2n}\,dm\right\}^{1/2} \left\{\int_{\partial\mathbb{U}} |g\circ\varphi|^2\,dm\right\}^{1/2} \\
&\le \|C_\varphi\| \left\{\int_{\partial\mathbb{U}} |f|^2\,|\varphi|^{2n}\,dm\right\}^{1/2}.
\end{aligned}$$

Since $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$, the last integral above converges to 0 as $n \to \infty$, which establishes that C_φ^* is SAT. □

It's easy to check that of all the notions of asymptotic toeplitzness we have dealt with here, only SAT fails to respect adjoints. In [1] Barría and Halmos give an example of an operator T that is SAT, but whose adjoint is *not*. Their example turns out to be the adjoint of the composition operator induced by the map $\varphi(z) = z^2$. Our final result generalizes their observation and, because adjoints preserve weak asymptotic toeplitzness, strengthens Proposition 2.1.

THEOREM 4.2 *If φ fixes the origin but is not a rotation, then C_φ^* is strongly asymptotically Toeplitz with asymptotic symbol $\equiv 0$.*

Proof The reproducing kernels $\{K_a : a \in \mathbb{U}\}$ introduced in the proof of Theorem 1.1 have linear span dense in H^2, and the operator norms $\|S^{*n}C_\varphi^*S^n\|$

are uniformly bounded, so it suffices to prove that

$$\lim_{n\to\infty} \|S^{*n} C_\varphi^* S^n K_a\| = 0 \tag{27}$$

for each $a \in \mathbb{U}$.

Let's begin by noting that

$$(S^n K_a)(z) = z^n K_a(z) = \frac{z^n}{1-\overline{a}z} = \frac{1}{\overline{a}^n}\left[K_a(z) - P_{n-1}(z)\right],$$

where $P_{n-1}(z) = \sum_{k=0}^{n-1} (\overline{a}z)^k$. This, along with the adjoint formula (2), yields

$$C_\varphi^* S^n K_a = \frac{1}{\overline{a}^n}[K_{\varphi(a)} - C_\varphi^* P_{n-1}]. \tag{28}$$

Since $\varphi(0) = 0$ the operator C_φ has lower triangular matrix with respect to the orthonormal basis $\{z^n\}_0^\infty$ for H^2. Thus C_φ^* has upper triangular matrix, i.e., $C_\varphi^* z^k$ is a polynomial of degree $\leq k$ for each non-negative integer k. In particular, $S^{*n} C_\varphi^* P_{n-1} = 0$, so by (28) above,

$$S^{*n} C_\varphi^* S^n K_a = \frac{1}{\overline{a}^n} S^{*n} K_{\varphi(a)} \tag{29}$$

for each non-negative integer n and each $a \in \mathbb{U}$.

To complete the proof note that $S^* K_b = \overline{b} K_b$ for any $b \in \mathbb{U}$, which, along with (29) yields

$$\|S^{*n} C_\varphi^* S^n K_a\| = \left|\frac{\varphi(a)}{a}\right|^n \|K_{\varphi(a)}\|$$

for each $n \geq 0$ and $a \in \mathbb{U}$. Since φ fixes the origin and is not a rotation, the Schwarz Lemma guarantees that $|\varphi(a)| < |a|$ for each $a \in \mathbb{U}$, which, along with the last equation, establishes (27). □

Theorems 4.2 and 3.2 show, in particular, that if a non-rotation φ fixes the origin and is inner, or more generally has radial limits of modulus one on a set of positive measure, then C_φ^* is strongly asymptotically Toeplitz but C_φ is not. As mentioned at the beginning of this section, The original example of Barría and Halmos is the special case $\varphi(z) = z^2$.

We do not know any non-rotational examples of composition operators whose adjoints *are not* strongly asymptotically Toeplitz. Perhaps they all are!

References

[1] J. Barría and P. R. Halmos, Asymptotic Toeplitz operators, *Trans. Amer. Math. Soc.* **273** (1982), 621–630.

[2] S. R. Bell and S. G. Krantz, Smoothness to the boundary of conformal maps, *Rocky Mountain J. Math.* **17** (1987) 24–40.

[3] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press 1995.

[4] R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press 1972 (2nd ed. Springer 1998).

[5] P. L. Duren, *Theory of H^p Spaces*, Academic Press 1970 Dover 2000.

[6] A. Feintuch, On asymptotic Toeplitz and Hankel operators, *Operator Theory, Advances and Applications,* **41**, Birkhaüser, Basel 1989, pp. 241–254.

[7] F. Jafari et al., editors, *Studies on Composition Operators,* Contemp. Math. Vol. 213, American Math. Soc. 1998.

[8] Y. Katznelson, *An Introduction to Harmonic Analysis,* Second ed., Dover, New York, 1976.

[9] S. G. Krantz, Lipschitz spaces, smoothness of functions, and approximation theory, *Exp. Math.* **3** (1983) 193–260.

[10] J. E. Littlewood, On inequalities in the theory of functions, *Proc. London Math. Soc.*, **23**, (1925), 481–519.

[11] J. E. Littlewood, *Lectures on the Theory of Functions,* Oxford University Press, 1944.

[12] M. Mateljević and M. Pavlović, The best approximation and composition with inner functions, *Michigan Math. J.,* **42**, (1995), 367–378.

[13] D. J. Newman and H. S. Shapiro, The Taylor coefficients of inner functions, *Michigan Math. J.,* **9**, (1962), 249–255.

[14] W. Rudin, *Real and Complex Analysis*, third edition, McGraw-Hill 1987.

[15] J. H. Shapiro, The essential norm of a composition operator, *Annals of Math.,* **125**, (1987), 375–404.

[16] J. H. Shapiro, *Composition Operators and Classical Function Theory,* Springer Verlag 1993.

[17] E. M. Stein, *Harmonic Analysis, Real variable methods, Orthogonality, and Oscillatory Integrals,* Princeton University Press 1993.