WHICH LINEAR-FRACTIONAL TRANSFORMATIONS INDUCE ROTATIONS OF THE SPHERE?

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Abstract. These notes supplement the discussion of linear fractional mappings presented in a beginning graduate course in complex analysis. The goal is to prove that a mapping of the Riemann sphere to itself is a rotation if and only if the corresponding map induced on the plane by stereographic projection is a linear fractional whose (two-by-two) coefficient matrix is unitary.

1. Spheres, points, and subspaces

1.1. **Point at infinity.** Recall that we have discussed two ways of "legitimizing" the "point at infinity" for the complex plane:

(a) *The Riemann Sphere* S^2 (cf. our textbook [S, pp. 8–11]). Here the idea is to map the extended plane $\hat{\mathbb{C}}$ onto the Riemann Sphere S^2 via the stereographic projection , making ∞ correspond to the north pole. Recall also that the stereographic projection
$$S^2 \backslash \{\text{North Pole}\} \to \mathbb{C}$$
is *conformal*.

(b) *Complex Projective space* $\mathbb{CP}^1$ (cf. [S, p. 25]). We regard this as the collection of one dimensional subspaces of $\mathbb{C}^2$, with the point $z \in \mathbb{C}$ identified with the subspace spanned by the column vector $[z, 1]^t$ (where the superscript "t" denotes "transpose"), and ∞ identified with the one spanned by $[1, 0]^t$.

1.2. **Notation.** Let $\hat{z}$ denote the one dimensional subspace of $\mathbb{C}^2$ spanned by the vector $[z, 1]^t$ if $z \in \mathbb{C}$, and let $\hat{\infty}$ be the subspace spanned by $[1, 0]^t$.

1.3. **Matrices and LFT's.** We have also made a connection between linear fractional transformations and matrices. This begins in a purely formal way by associating each LFT $\varphi(z) = \frac{az+b}{cz+d}$ with the two-by-two complex nonsingular matrix $[\varphi] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, noting that actually $[\varphi]$ should not just stand for one matrix, but for the one-parameter family: all the multiples

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of the above matrix by non-zero complex numbers (reflecting the fact that if we multiply all the coefficients of a LFT by the same non-zero complex number, we don't change the transformation).

We connect the action of the LFT φ on S^2 with that of any of its matrices on $\mathbb{C}^2$ by noting that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} az+b \\ cz+d \end{bmatrix},$$

so making the more proper interpretation of $[\varphi]$ as the collection of *all multiples* of any matrix whose coefficients are those of φ (including zero now), we have—in the notation of §1.2—$[\varphi](\hat{z}) = \widehat{\varphi(z)}$, i.e., for any $z \in \hat{\mathbb{C}}$:

> *The one dimensional subspace you get by allowing the collection of matrices that represent φ (now including the zero-matrix) to act on $\hat{z}$ is just the (one dimensional) subspace spanned by $\widehat{\varphi(z)}$.*

1.4. **Corollary.** *Suppose φ is a LFT and $z \in \hat{\mathbb{C}}$. Then the following are equivalent:*

(a) *z is a fixed point of φ.*
(b) *$\hat{z}$ is an eigenvector of $[\varphi]$ (i.e. of each matrix that represents φ).*
(c) *$[\varphi]$ fixes the subspace $\hat{z}$.*

1.5. **Matrices and maps of the sphere.** We've also discussed, via some homework problems, the fact that LFT's can be viewed as conformal maps of the Riemann sphere. Thus we can view two-by-two complex nonsingular matrices as acting *conformally on the Riemann Sphere.* Perhaps most important of the complex matrices are the *unitary* ones—the ones whose adjoints (i.e., conjugate transposes) are their inverses. The question we wish to pose is:

> *What do two-by-two unitary matrices do to S^2?*

1.6. **Example.** The LFT $\varphi(z) = 1/z$ is given by a unitary matrix (zero on main diagonal, 1 on the cross-diagonal). It has fixed points ± 1, and we saw in a homework problem that its action on S^2 can be viewed as a composition of two reflections in orthogonal planes that contain the real axis. So it's reflection in the intersection of these planes, i.e., in the real axis, i.e., it's *rotation by* $180°$ *about the real axis!*

2. Unitary matrices

For an $n \times n$ complex matrix A let A^* denote its adjoint. As mentioned above, "A unitary" means "A is invertible, with inverse equal to A^*." It's easy to see that A is unitary iff its columns form an orthonormal basis for $\mathbb{C}^n$. (Sketch of proof: The (i, j) element of the product of two matrices is the dot product (no complex conjugates this time) of the i-th row of the first matrix with the j-th column of the second one.)

Here a set of vectors in $\mathbb{C}^n$ is said to be "orthonormal" if its elements all have norm one, and are "pairwise orthogonal" with respect to the inner product of $\mathbb{C}^n$ (defined like the "real dot product," but with elements of the second vector conjugated; see [A, Chapter 6]). One checks easily that for any $n \times n$ complex matrix A,

$$\langle Av,\, w\rangle = \langle v,\, A^* w\rangle \qquad \forall v, w \in \mathbb{C}^n,$$

and from this it's easy to check that unitary matrices preserve norms and inner products:

2.1. **Proposition.** *If A is an $n \times n$ unitary matrix, then for any pair of vectors $v, w \in \mathbb{C}^n$ we have $\langle Av,\, Aw\rangle = \langle v,\, w\rangle$. In particular $\|Av\| = \|v\| \quad \forall v \in \mathbb{C}^n$.*

For more details on this, and other aspects of unitary matrices and operators, see [A, Chapter 7] (where the term "isometry" is used instead of "unitary operator").

In what follows we take a path that emphasizes the action of a unitary matrix as a linear transformation on $\mathbb{C}^n$, and prove the spectral theorem for such matrices. For an alternate, more computational, way of proceeding that emphasizes the two-by-two case and does not mention linear transformations see §4 below. Either of these routes will provide, for each unitary matrix A, a crucial decomposition of $\mathbb{C}^2$ into orthogonal eigensubspaces of A.

For a subspace M of $\mathbb{C}^n$ let $M^\perp$ denote the "orthogonal complement" of M, i.e. the set of vectors in $\mathbb{C}^n$ orthogonal to M. It's easy to check that $(M^\perp)^\perp = M$, and that the direct sum of $M^\perp$ and M is all of $\mathbb{C}^n$. A special property of unitary matrices is this, where we now view matrices as linear transformations acting on $\mathbb{C}^n$ (viewed as the space of all n-dimensional column vectors):

2.2. **Proposition.** *If M is a subspace of $\mathbb{C}^n$ and A is an $n \times n$ unitary matrix with $A(M) \subset M$, then $A(M^\perp) \subset M^\perp$.*

In other words, for a unitary matrix, every invariant subspace *reduces* the linear transformation represented by the matrix, in the sense that $\mathbb{C}^n$ can be decomposed into the orthogonal direct sum of two nontrivial subspaces that are invariant [1] for the transformation.

Proof. Suppose $v \in M^{\perp}$; we want to show $Av \in M^{\perp}$. Let w be any vector in M. Then

$$\langle Av, w\rangle = \langle v, A^*w\rangle = \langle v, A^{-1}w\rangle.$$

Now A is invertible, so $A(M)$ has the same dimension as M, and since it is contained in M it must *equal* M (finite dimensionality is crucial here!). Thus $A^{-1}M = M$, hence in the above inner-product equation $A^{-1}w \in M$, so the last inner product is zero. Thus $Av \in M^{\perp}$, as desired. □

This sets the stage for the most important result about unitary matrices:

2.3. **The Spectral Theorem for Unitary Matrices.** *For each unitary $n \times n$ matrix A there is an orthonormal basis for $\mathbb{C}^n$ consisting entirely of eigenvalues of A.*

Proof. The characteristic equation $\det(A) - \lambda I = 0$ is an n-th degree polynomial in λ with complex coefficients, so it has a complex solution λ_1 (by the Fundamental Theorem of Algebra). Thus the matrix $A - \lambda_1 I$ is singular, so it left-multiplies some non-zero vector v_1 to the zero-vector. Thus v_1 is an eigenvector of A (with λ_1 the corresponding eigenvalue).

The subspace of $\mathbb{C}^n$ spanned by the eigenvector v_1 is invariant under A, hence, by the last Proposition, so is its orthogonal complement M_1. If $M_1 = \{0\}$ then $n = 1$ and we are done. Otherwise, restrict the linear transformation induced by A to M_1 and repeat the "characteristic polynomial" argument of the last paragraph, with the operator (represented by the matrix) A replaced by its restriction to M_1. This produces an eigenvector v_2 that lies in M_1, and so is orthogonal to v_1. The subspace of C^n spanned by v_1 and v_2 is again invariant for A; if it's all of $\mathbb{C}^n$ then we're done (with $n = 2$). Otherwise the orthogonal complement M_2 this subspace consists of more than the zero-vector, and is also invariant under A, so as before *it* contains an eigenvector v_3 for A, necessarily orthogonal to v_1 and v_2. Keep going until you get n of these! □

2.4. **Remark.** The norm-preserving property guarantees that for a unitary matrix, each eigen*value* must have modulus one. The Spectral Theorem above says that every unitary matrix has n eigenvalues (multiplicity counted). Thus *the determinant of any unitary matrix has modulus one* (it's the product of the eigenvalues). This can be seen more directly by noting

[1] To say a subspace M of a vector space is "invariant" for a linear transformation A means that $AM \subset M$. The proof of Proposition 2.2 shows that for *unitary* transformations A we have $AM = M$ for each invariant subspace.

that if A is unitary and $\Delta = \det A$, then $\overline{\Delta} = \det A^* = \det A^{-1} = 1/\Delta$, hence $1 = \overline{\Delta}\Delta = |\Delta|^2$.

3. Unitary matrices and LFT's

Suppose now that an LFT φ is represented by a unitary two-by-two matrix A. By the Spectral Theorem, A has orthogonal eigenvectors v_1 and v_2 in $\mathbb{C}^2$. These correspond to points z_1 and z_2 of the extended plane $\hat{\mathbb{C}}$, and by Corollary 1.4 we know that these two points are *fixed points* of φ. What does the *orthogonality* of their corresponding $\mathbb{C}^2$-subspaces say about these points?

3.1. **Lemma.** *For z_1 and z_2 in $\hat{\mathbb{C}}$, the following are equivalent:*

(a) *The $\mathbb{C}^2$-subspaces $\hat{z_1}$ and $\hat{z_2}$ are* orthogonal.
(b) *The stereographic projections z_1^* and z_2^* are* antipodal *(i.e., $z_1^* = -z_2^*$ in $\mathbb{R}^3$).*
(c) $z_2 = -1/\overline{z_1}$.

Proof. Recall that for $z \in \mathbb{C}$ we've defined $\hat{z}$ to be the subspace spanned by the vector $\tilde{z} \stackrel{\text{def}}{=} [z,1]^t$, so if z_1 and z_2 are finite, then $\langle \tilde{z_1}, \tilde{z_2} \rangle = z_1\overline{z_2} + 1$, which establishes the equivalence of (a) and (c) for finite z's. If (say) $z_1 = \infty$ then $\hat{z_1}$ is the subspace spanned by the vector $[1,0]^t$, hence $\hat{z_2} \perp \hat{z_1}$ iff $\hat{z_2}$ is spanned by $[0,1]$, i.e. iff $z_2 = 0$. Thus (a) and (c) are equivalent in any case. The equivalence of (c) and (b) was noted in a previous homework problem (see Problem Set III). □

3.2. **Corollary.** *If a LFT φ has a unitary matrix then φ has two distinct fixed points in $\hat{\mathbb{C}}$, and these correspond, via stereographic projection, to antipodal points of the Riemann Sphere.*

What about the distance-preserving property of unitary matrices? For a two-by-two unitary does this get reflected somehow in the action of the corresponding LFT on the extended plane? The next result says "yes"!

3.3. **Theorem.** *If φ is a LFT given by a unitary matrix, then φ preserves the spherical metric, i.e., $\rho(\varphi(z), \varphi(w)) = \rho(z,w)$ for every pair of points $z, w \in \hat{\mathbb{C}}$.*

Proof. Let's continue with the notation $\tilde{z} = [z,1]^t$ for $z \in \mathbb{C}$, and $\tilde{\infty} = [1,0]^t$. Then the formula for spherical metric [S, page 11] can be rewritten (for z, w finite):

$$\rho(z,w) \stackrel{\text{def}}{=} \frac{2|z-w|}{\sqrt{|z|^2+1}\,\sqrt{|w|^2+1}} = \frac{2\|\tilde{z}-\tilde{w}\|}{\|\tilde{z}\|\|\tilde{w}\|} \tag{1}$$

Note also that

$$\widetilde{\varphi(z)} \stackrel{\text{def}}{=} \begin{bmatrix} \frac{az+b}{cz+d} \\ \\ 1 \end{bmatrix} = \frac{1}{cz+d} \begin{bmatrix} az+b \\ \\ cz+d \end{bmatrix},$$

that is,

$$\widetilde{\varphi(z)} = \frac{1}{cz+d}\, A\tilde{z} \qquad \forall z \in \hat{\mathbb{C}}. \tag{2}$$

Now let's put these observations together in an elementary calculation:

$$\begin{aligned} \rho(\varphi(z), \varphi(w)) &= \frac{2\|\widetilde{\varphi(z)} - \widetilde{\varphi(w)}\|}{\|\widetilde{\varphi(z)}\|\|\widetilde{\varphi(w)}\|} = \frac{2\|\frac{A\tilde{z}}{cz+d} - \frac{A\tilde{w}}{cw+d}\|}{\|\frac{A\tilde{z}}{cz+d}\|\|\frac{A\tilde{w}}{cw+d}\|} \\ &= 2\,\frac{\|A\tilde{z}(cw+d) - A\tilde{w}(cz+d)\|}{\|A\tilde{z}\|\|A\tilde{w}\|} \\ &= 2\,\frac{\|A\{\tilde{z}(cw+d) - \tilde{w}(cz+d)\}\|}{\|A\tilde{z}\|\|A\tilde{w}\|}\,. \end{aligned}$$

Since unitary matrices preserve norms (Proposition 2.1), this yields

$$\rho(\varphi(z), \varphi(w)) = 2\,\frac{\|\tilde{z}(cw+d) - \tilde{w}(cz+d)\|}{\|\tilde{z}\|\|\tilde{w}\|} \tag{3}$$

Let's work on the numerator of the fraction on the right-hand side of the last expression:

$$\|\tilde{z}(cw+d) - \tilde{w}(cz+d)\| = \left\| \begin{bmatrix} d(z-w) \\ c(w-z) \end{bmatrix} \right\| = |z-w| \left\| \begin{bmatrix} d \\ c \end{bmatrix} \right\|$$

Now the columns of the unitary matrix A^* have norm 1 in $\mathbb{C}^2$, so the last calculation yields:

$$\|\tilde{z}(cw+d) - \tilde{w}(cz+d)\| = |z-w| = \|\tilde{z} - \tilde{w}\|.$$

Upon putting this together with (3) we obtain:

$$\rho(\varphi(z), \varphi(w)) = 2\,\frac{\|\tilde{z} - \tilde{w}\|}{\|\tilde{z}\|\|\tilde{w}\|} = \rho(z, w),$$

as promised. If one of z, w is ∞ the proof is similar, and easier. Alternately, one could "just take limits" to get to infinity; I leave the details to you. □

So far we've proved that LFT's induced by unitary matrices have these properties, when viewed as 1-1 maps taking the Riemann Sphere onto itself:

- They are conformal (from conformality of LFT's on the plane, and of the stereographic projection).
- They have antipodal fixed points.
- They preserve distances.

Now rotations of the sphere have the above three properties, and it turns out—as we prove in §5 below—these are the *only* such mappings (see also [N, page 36]). This, along with the work done here, proves one half of the result these notes have been aiming for:

3.4. **Theorem.** *A mapping of S^2 into itself is a rotation if and only if it is induced, via stereographic projection, by a unitarily-represented LFT.*

Proof. We've already shown (modulo the work of §5) that unitarily-represented LFTs induce rotations of S^2. Suppose conversely that we have a rotation of the sphere. If the axis of this rotation is the polar one (the line through the north and south poles), then it's clear that the map induced on the plane via stereographic projection is a rotation: $\varphi(z) \equiv \omega z$ for some complex number ω of unit modulus. This φ is an LFT represented by the unitary matrix $\begin{bmatrix} \omega & 0 \\ 0 & 1 \end{bmatrix}$.

Suppose the axis of rotation is not the polar axis. Then the induced map φ of $\hat{\mathbb{C}}$ fixes points z_1 and z_2 of $\mathbb{C}\backslash\{0\}$. Write $z_1 = -b/a$ where $|a|^2 + |b|^2 = 1$ and consider the LFT $\psi(z) = \frac{az+b}{-\bar{b}z+\bar{a}}$, which takes z_1 to the origin. It is represented by the matrix $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$, which has orthonormal columns, hence is unitary. Thus ψ induces a rotation of S^2. Moreover the map $\psi\circ\varphi\circ\psi^{-1}$ fixes the origin, and its corresponding sphere map is a composition of rotations, hence is itself a rotation (that a composition of rotations is a rotation is another geometric result we won't prove here). Just as in the previous paragraph, $\psi \circ \varphi \circ \psi^{-1}$ is a rotation of the plane: $z \to \omega z$ for some unimodular complex number ω, hence $\varphi(z) \equiv \psi^{-1}(\omega\psi(z))$. This exhibits φ as the composition of three unitarily-represented LFT's, hence it is itself a LFT that is unitarily represented by the matrix

$$\underbrace{\begin{bmatrix} \bar{a} & -b \\ \bar{b} & a \end{bmatrix}}_{[\psi^{-1}]} \underbrace{\begin{bmatrix} \omega & 0 \\ 0 & 1 \end{bmatrix}}_{[z\to\omega z]} \underbrace{\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}}_{[\psi]}.$$

This completes the proof. □

3.5. **Remark on stereographic conformality.** As a curious byproduct of this connection between linear fractional transformations and rotations of the sphere we get an easy proof of the conformality of the stereographic projection. It's intuitively clear that two smooth plane curves intersecting at the origin have stereographic images that intersect at the south pole of S^2 at the same angle, and with the same sense (this is perhaps most easily visualized if we think of the complex plane as being the plane tangent to S^2 at the south pole, rather than our usual interpretation of $\mathbb{C}$ as the equatorial plane). If the original curves intersect at any other point p of the plane, we may, as in the above proof of Theorem 3.4, move p to the origin by means of a unitarily represented LFT, which is conformal at p and induces a rotation of S^2 (also conformal, of course). Thus conformality at p follows from conformality at the origin. □

4. Appendix I: two-by-two unitaries without linear transformations.

Let A be a two-by-two matrix. In the first paragraph of §2 we observed that A is unitary iff its columns form an orthonormal basis for $\mathbb{C}^2$. Thus (as we observed in the proof of Theorem 3.4 above) every two-by-two matrix of the form:

$$A = \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}, \tag{4}$$

with $|a|^2 + |b|^2 = 1$, is unitary (with determinant 1).

4.1. **Theorem.** *Every two-by-two unitary matrix with determinant 1 has the form (4), with* $|a|^2 + |b|^2 = 1$

Proof. Suppose $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is unitary with determinant 1. Then

$$\begin{aligned} a\overline{b} + c\overline{d} &= 0 \\ ad - cb &= 1 \end{aligned}$$

where the first equation expresses the orthogonality of the columns and the second the fact that the determinant is 1. In other words the system of linear equations

$$\begin{aligned} \overline{b}x + \overline{d}y &= 0 \\ dx - by &= 1 \end{aligned}$$

has solution $x = a, y = c$. Now the determinant of the coefficients of this system is $-(|b|^2 + |d|^2) = -1$ (the term in parentheses is the norm of the second column of the matrix, which is 1). So the system has a *unique* solution which, by Cramer's Rule (for example), turns out to be $x = \overline{d}$, $y = -\overline{b}$, hence $a = \overline{d}$ and $b = -\overline{b}$, so our matrix has the desired form. □

Now we can show that the fixed points of a unitary-induced LFT correspond to antipodal points of the Riemann sphere.

4.2. **Theorem.** *Suppose φ is a LFT whose matrix is unitary. Then φ has two distinct fixed points in $\hat{\mathbb{C}}$. If these are z_2 and z_2 then $z_2 = -1/\overline{z_1}$.*

Remark. As we remarked in proving Lemma 3.1 it follows from a homework problem that the stereographic mapping of these points onto the Riemann sphere are antipodal, and from from that Lemma we also know that the subspaces $\hat{z_1}$ and $\hat{z_2}$ of C^2 corresponding to these points are orthogonal eigensubspaces. Thus we have the following special case of Theorem 2.3 (the Spectral Theorem): *Every two-by-two unitary matrix has an orthogonal pair of eigenvectors.*

Proof of Theorem. We may assume the matrix of coefficients of φ has determinant 1. By Theorem 4.1 this matrix has the form (4) with $|a|^2 + |b|^2 = 1$, i.e.,

$$\varphi(z) = \frac{az + b}{-\bar{b}z + \bar{a}} \tag{5}$$

Now $\varphi(0) = 0$ iff $b = 0$, so $\varphi(z) = (a/\bar{a})z$, a rotation about the origin. Therefore ∞ is the other fixed point.

So suppose $b \neq 0$. Then it's easy to see that the fixed points of φ satisfy the quadratic equation $p(z) = 0$, where

$$p(z) = \bar{b}z^2 + (a - \bar{a})z + b.$$

A quick calculation shows that

$$\bar{z}^2 p(-\frac{1}{\bar{z}}) = \overline{p(z)},$$

hence $z(\neq 0)$ satisfies $p(z) = 0$ iff the same is true of $-1/\bar{z}$. Thus if z_1 is a fixed point of φ, then so is $z_2 = -1/\overline{z_1}$. □

It remains to give a proof of Theorem 3.3, the spherical distance-preserving property of unitary-induced LFT's. This follows from the representation (4) (with $|a|^2 + |b|^2 = 1$) via a direct calculation. I leave this one to you.

5. Appendix II: Conformal isometries are rotations

To complete our proof that unitarily induced LFTs correspond, via stereographic projection, to rotations of the sphere, we need to show that every conformal isometry of the sphere with antipodal fixed points is a rotation. So let T be such a mapping.

Step 1: *Extension to* $\mathbb{R}^3$. For $x \in \mathbb{R}^3 \backslash \{0\}$ let

$$x' = \frac{x}{\|x\|} \quad \text{and} \quad T(x) \stackrel{\text{def}}{=} \|x\| T(x').$$

Define $T(0) = 0$.

Step 2: *Preservation of angles.* Suppose first that $x, y \in S^2$ (i.e., they're unit vectors). Then because T is an isometry on S^2: $\|T(x)-T(y)\| = \|x-y\|$. Upon squaring both sides of this equation,using the fact that the norm-squared of a vector equals the dot product of that vector with itself, and then the isometric property again, we see quickly that $T(x) \cdot T(y) = x \cdot y$. Thus T *preserves angles between vectors.*

Now suppose x and y are any non-zero vectors in $\mathbb{R}^3$. Then using the definitions and notation of Step 1:

$$T(x) \cdot T(y) = \|x\|\|y\|T(x') \cdot T(y') = \|x\|\|y\|x' \cdot y' = x \cdot y.$$

So, at least for non-zero vectors, T preserves dot products. It's trivial to see that the same is true if one of x, y is zero (since $T0 = 0$ by definition).

Step 3: *Linearity.* Use the "preservation of dot products" proved in Step 2 to check (by expressing norm-squared in terms of dot product) that for any $x, y \in \mathbb{R}^3$ and $a \in \mathbb{R}$:

$$\|T(x) - T(y) - T(x-y)\| = 0 \quad \text{and} \quad \|aT(x) - T(ax)\| = 0.$$

Step 4: *Finale.* So now our extended T is a linear isometry on $\mathbb{R}^3$, i.e., *an orthogonal linear transformation.* Since it also has antipodal fixed points, it fixes every point of the line through these fixed points. i.e., this line is an eigensubspace corresponding to the eigenvector 1. The plane orthogonal to this line is an invariant subspace for T (same argument we used in the proof of the Spectral Theorem for Unitary Matrices—Theorem 2.3), and it's easy to check that conformality plus orthogonality requires the restriction of T to this plane to be a rotation. Thus T, acting on $\mathbb{R}^3$, is a rotation about the line through the antipodal fixed points of the original sphere-map. □

References

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