Non-measurable sets

1 Introduction

The purpose of these notes is to show how our original construction of a nonmeasurable set can be refined to yield a set with even more remarkable properties. I'll also show how our original construction leads to results that are even more paradoxical than we might at first have thought.

We work on the half-open interval $G = [0, 1)$ which we regard as a commutative group with addition mod 1. In what follows the operations of addition, subtraction, and multiplication on elements of G will be denoted by the usual symbols, but the results of these operations are always understood to be reduced modulo 1.

The goal here is to produce a non-measurable subset NM of G that has outer measure 1 and inner measure zero (in the sense that every measurable subset has measure zero). Necessarily such a set is "universally non-measurable" in the sense that its intersection with any measurable set of positive measure is non-measurable.

To see why this is so, note first that NM' also has outer measure 1 and inner measure zero. To prove the former assertion, suppose E is measurable and contains NM'. Then E' is measurable and contained in NM, so E' has measure zero. Therefore (by measurability) E has measure one. The statement about inner measure is proved similarly.

Now suppose E is a measurable subset of G whose intersection with NM is measurable. Then $E \cap NM' = E \backslash (E \cap NM)$ is measurable, so because both NM and NM' have inner measure zero, both $E \cap NM$ and $E \cap NM'$ have measure zero, hence so does their union E. Thus any set whose intersection with NM is measurable has measure zero. Equivalently, the intersection of a set of positive measure with NM must be nonmeasurable. □

2 Dense Subgroups of G

The rationals in G form a subgroup, and they are dense in G. The main result of this section shows that this density can be viewed within a larger context.

2.1 Theorem. *Every infinite subgroup of G is dense.*

Proof. Suppose H is a subgroup of G that has infinitely many elements. Let I be any interval in G. Our goal is to show that some element of H lies in I. Fix a positive integer k with $1/k <$ length of I. Let $h_1 < h_2 < \ldots < h_{k+1}$ be $k + 1$ distinct elements of G, arranged in increasing order. Then for some index j we must have $d = h_{j+1} - h_j < 1/k$. Thus $nd \in I$ for some positive integer n. Now $d \in H$, hence $nd \in H$, and we are done. □

2.2 Exercise. let $\mathcal{T}$ denote the unit circle with complex multiplication, taken in the metric of $\mathbb{R}^2$. Show that every infinite subgroup of $\mathcal{T}$ is dense in $\mathcal{T}$.

2.3 Exercise. Show that for every irrational number ξ:

(a) The sequence $\{e^{2\pi in\xi} : n \in \mathbb{Z}\}$ is dense in the unit circle.

(b) The sequences $\{\sin(2\pi n\xi) : n \in \mathbb{N}\}$ and $\{\cos(2\pi n\xi) : n \in \mathbb{N}\}$ are dense in $[-1, 1]$.

3 Difference sets

If $A \subset G$ we define

$$A - A \stackrel{\text{def}}{=} \{a - b : a, b \in A\}.$$

Note that for $x \in g$:

$$x \in A - A \iff A \cap (A + x) \neq \emptyset.$$

3.1 Theorem. *If E is a measurable subset of G and E has positive measure, then $E - E$ contains an interval.*

Proof. By the Lebesgue Density Theorem there is a point $0 \neq d \in E$ such that

$$\frac{\mu(E \cap I(d, h))}{2h} > 3/4, \tag{1}$$

for all sufficiently small $h > 0$, where $I(d, h) = (d - h, d + h)$. Now fix a "sufficiently small" h that is also small enough that both $d - h$ and $d + 2h$ lie in G (without carrying out any reductions mod 1).

Claim: *The interval $[0, h)$ lies in $E - E$.*

To see this, fix $x \in (0, h)$ (there's no doubt that 0 lies in every difference set). We'll show that $(E + x) \cap E \neq \emptyset$. To simplify notation, let $I = I(d, h)$ and set $F = E \cap I$. Then because $0 < x < h$, both F and $F + x$ lie in the interval $[d - h, d + 2h]$. By (1) both F and $F+x$ have measure $> 3h/2$, so if they were disjoint, their union—which is contained in $[d-h, d+2h]$—would have measure $> 2 \times 3h/2 = 3h$, which contradicts the fact that the containing interval only has measure $3h$. Thus F and $F + x$ have nontrivial intersection, hence the same is true of their supersets E and $E + x$. $\square$

4 Construction of NM

Fix an irrational number $\xi \in G$ and let

$$A = \{n\xi : n \in \mathbb{Z}\} \quad \text{and} \quad B = \{2n\xi : n \in \mathbb{Z}\}$$

Then A and B are subgroups of G, both infinite because of the irrationality of ξ. Therefore by Theorem 2.1, both A and B are dense in G. Now $A = B \cup C$, a disjoint union, where

$$C = \{(2n+1)\xi : n \in \mathbb{Z}\}.$$

C is also dense in G since it equals $B + \xi$, and B is dense.

Let NM_0 be the set you get by choosing one element from each of the cosets of G modulo A [1]. In other words, G is partitioned into equivalence classes by the equivalence relation

$$g_1 \equiv g_2 \iff g_1 - g_2 \in A,$$

and we are choosing one element from each equivalence class. Then, as we saw in our original construction of a non-measurable set, not only is G the disjoint union of sets $g + A$ as g runs through NM_0, *it is also the* countable *disjoint union of the sets* $a + NM_0$ *as* a *runs through* A. Just as for our original example, NM_0 has inner measure zero. For if F is a subset of NM_0 that is measurable, then G contains the countable disjoint union f sets $F + a$ as a runs through A. Since all these sets have the same measure, translation-invariance (modulo 1) of Lebesgue measure (on G), F must have measure zero, else G would have infinite measure, an absurdity.

Here is another proof that uses the translation-invariance of Lebesgue measure more subtlely—can you see where it shows up? NM_0 is constructed so that its difference with itself contains no element of A other than 0. Thus the same will be true of any measurable subset F of NM_0. Since A is dense, $F - F$ cannot contain an interval, and so by Theorem 3.1 F cannot have positive measure.

We use NM_0 to construct the non-measurable set that simultaneously has inner measure zero and outer measure 1. We claim that

$$NM \stackrel{\text{def}}{=} NM_0 + B = \bigcup_{b \in B} (NM_0 + b).$$

has the desired property. We phrase these properties like this:

4.1 Theorem. *Every measurable subset of NM has measure zero. Every measurable superset of NM has measure 1.*

Proof. To see that NM has inner measure zero, note that

$$NM - NM = NM_0 - NM_0 + B$$

and since $NM_0 - NM_0$ does not contain any non-zero element of A, one checks easily that $NM - NM$ does not intersect C. Since C is dense, $NM - NM$ contains no interval, so as above, any measurable subset of NM must have measure zero, which establishes the desired "inner measure zero" property of NM.

For the "outer measure one" property note that the complement of NM in G is just the translate $NM + \xi$, which—by the translation-invariance of Lebesgue measure—also has inner measure zero. Thus if F is a measurable superset of NM, then F' is a

[1] We use the Axiom of Choice here

measurable subset of NM', and so has measure zero, so F itself has measure 1. Thus NM has outer measure 1 as advertised. □

The following property of NM was noted in the Introduction:

4.2 Corollary. *NM is "universally non-measurable" in the sense that its intersection with any measurable subset of G having positive measure is non-measurable.*

4.3 Corollary. *Every measurable subset of $\mathbb{R}$ having positive measure contains a non-measurable subset.*

PROOF. Exercise. □

5 Paradoxical decompositions

Suppose we had just defined Lebesgue outer measure on G, and were asking if that definition gave a reasonable extension of the notion of "length" to all subsets of G. The set NM and its complement $NM + \xi$ show that this approach needs refinement, since they give a "paradoxical" decomposition of G—which has length one—into two disjoint subsets, both of which also, according to our new definition, have "length one."

The set NM_0 (or equally effectively, our original nonmeasurable set) gives another kind of paradoxical decomposition that is, in some sense, even stranger. Recall that G is the disjoint union of sets $NM_0 + a$ where a runs through the dense subgroup A generated by the irrational number ξ. Let's enumerate the elements of a as $a_0 = 0, a_1, a_2, \ldots$, and write $M_j = NM_0 + a_j$ (in particular $M_0 = NM_0$). Consider the pairwise disjoint sets $M_0, M_2, M_4, \ldots$, whose union is our universally nonmeasurable set NM. Now observe that M_2 can be translated onto M_1, M_4 can be translated onto M_2, M_6 onto M_3, etc. In other words, the disjoint pieces $\{M_{2n}\}_0^\infty$ can be individually translated so they can be reassembled into all of G!

Similarly, the disjoint pieces $\{M_{2n+1}\}_0^\infty$ can be individually translated and reassembled into all of G. Thus G can be broken into a countable collection of pairwise disjoint which can be appropriately translated and reassembled into *two* copies of G!

It's perhaps better to think of G as being the unit circle here, in which case the translations become rotations: isometric motions of the circle. Even more amazing results were proved in the period 1915–25 by Hausdorff, Banach, and Tarski. Here is the most famous of these:

The Banach-Tarski Paradox. *The unit ball of $\mathbb{R}^3$ can be decomposed into a finite collection of pairwise disjoint sets which can be reassembled by rigid motions into a ball of radius 2.*

In fact, Hausdorff, Banach, and Tarski showed that given any two bounded subsets A and B of $\mathbb{R}^3$ that have nonempty interior, you can cut A into finitely many pairwise disjoint sets which you can then rearrange by rigid motions to form B. This has to be one of the strangest results ever proved by mathematicians!

For more on the Banach-Tarski paradox and its implications for mathematics, see Stan Wagon's book *The Banach-Tarski Paradox,* Cambridge Univ. Press 1985, and a little (4 page) article by Frank Wikström titled *the Banach-Tarski Theorem* which is available on the Web at

{\tt http://abel.math.umu.se/~frankw/articles/index.html}

(or you can get a copy from me).