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MEAN GROWTH OF KOENIGS EIGENFUNCTIONS

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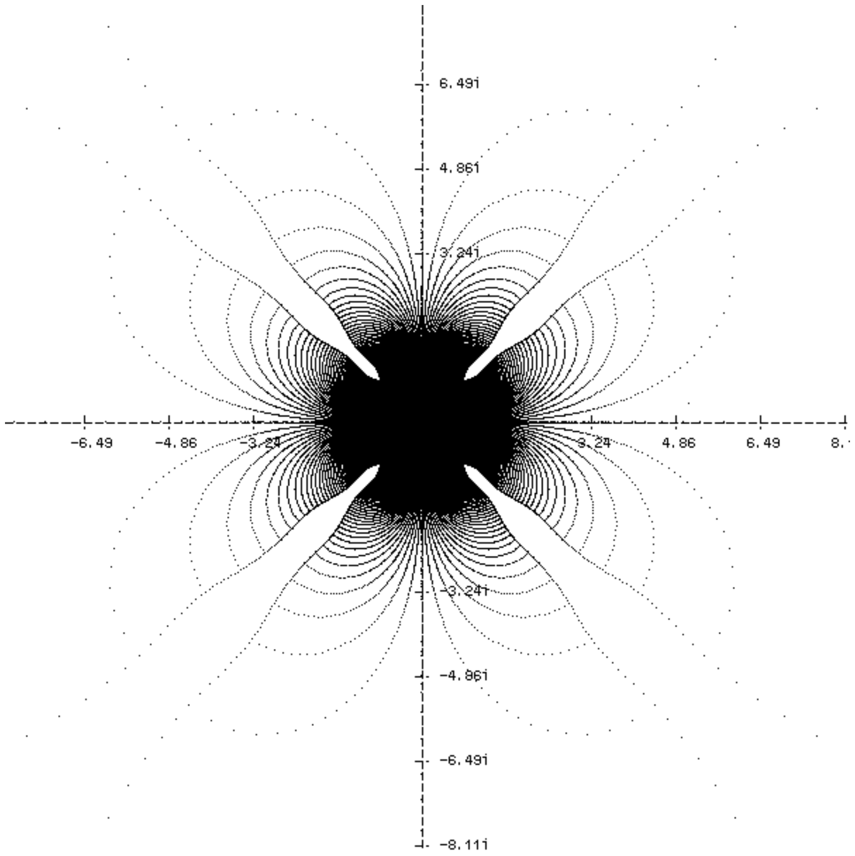

FIGURE 1. The shaded region is the image of the unit disk under the Koenigs eigenfunction σ for $\varphi(z) = z/(2 - z^4)$, the four fixed points of φ on the unit circle corresponding to points where σ has infinite radial limit. The function σ belongs to the Hardy space H^p if and only if $0 < p < \log(5)/\log(2)$.

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1. Introduction

Let U be the open unit disk in the complex plane $\mathbf{C}$ and let $\varphi : U \to U$ be a holomorphic function fixing a point a in U. To avoid trivial situations, we assume $0 < |\varphi'(a)| < 1$. For n a positive integer, let φ_n denote the n-th iterate of φ so that

$$\varphi_n = \varphi \circ \varphi \circ \cdots \circ \varphi, \qquad n \text{ times.}$$

Over a century ago, Koenigs ([15]) showed that if $f : U \to \mathbf{C}$ is a nonconstant holomorphic solution to Schroeder's functional equation,

$$f \circ \varphi = \lambda f,$$

then there exist a positive integer n and a constant c such that $\lambda = \varphi'(a)^n$ and $f = c\sigma^n$, where for each $z \in U$,

$$\sigma(z) := \lim_{n\to\infty} \frac{\varphi_n(z) - a}{\varphi'(a)^n}. \tag{1.1}$$

We call the function σ defined by (1.1) the *Koenigs eigenfunction* of φ; it is the unique solution of

$$\sigma \circ \varphi = \varphi'(a)\sigma$$

having derivative 1 at a.

While the local study of Koenigs eigenfunctions plays a fundamental role in complex dynamics, the study of how global properties of φ influence those of σ is just beginning (see [2, 18, 19, 28]). In this paper we obtain results showing how properties of φ influence the growth and mean growth of $\sigma(z)$ as $|z| \to 1^-$, mean growth being measured by Hardy-space membership.

Let f denote an analytic function on U. Recall that f belongs to the Hardy space H^p for some $0 < p < \infty$ provided that the integral means

$$M_p(r, f) := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}$$

are uniformly bounded for $r \in [0, 1)$, in which case we define

$$\|f\|_p = \sup\{M_p(r, f) : 0 \le r < 1\}.$$

Recall that f belongs to H^∞ provided it is bounded, in which case we define

$$\|f\|_\infty = \sup\{|f(z)| : z \in U\}.$$

If $1 \le p \le \infty$ then $\|\cdot\|_p$ is a norm that makes H^p into a Banach space, while if $0 < p < 1$ then $\|\cdot\|_p^p$ is a "p-homogeneous norm" that makes H^p into a "p-Banach space" (see [9, Section 3.2]). We note that information about mean-growth yields information about maximum growth because for each $f \in H^p$, we have

$$|f(z)| \le \frac{2^{1/p}\|f\|_p}{(1 - |z|)^{1/p}}.$$

([9, p. 36]). We also note that every holomorphic mapping $\varphi : U \to U$ induces a bounded composition operator on H^p (see, e.g., [7] or [26]), which we'll denote by C_φ :

$$C_\varphi f = f \circ \varphi.$$

The Koenigs eigenfunction σ for φ need not belong to any of the H^p spaces; for example, we show this is the case when φ is an inner function (see Proposition 3.2). On the other hand, when φ is univalent, σ must also be univalent and hence must belong to H^p for at least $0 < p < 1/2$ (see Proposition 2.3). Our main theorem provides a sufficient condition for σ to belong to H^p. The condition is expressed in terms of the essential spectral radius of the composition operator $C_\varphi : H^2 \to H^2$; however, as we will shortly explain, it may be expressed in purely function-theoretic terms.

Main Theorem. *Let $0 < p < \infty$. If $|\varphi'(a)|^{p/2}$ exceeds the essential spectral radius of $C_\varphi : H^2 \to H^2$, then the Koenigs eigenfunction σ for φ belongs to H^p.*

There is evidence that the preceding sufficient condition is also necessary for σ to belong to H^p. Theorem 4.7 below establishes necessity when φ is analytic on the closed disk. Furthermore, Pietro Poggi-Corradini has recently shown the sufficient condition of the Main Theorem is necessary when φ is univalent [19].

In proving the Main Theorem, we devote the bulk of our effort to establishing that the essential spectral radius of $C_\varphi : H^p \to H^p$, denoted by $r_e(C_\varphi|_{H^p})$, is less than or equal to $[r_e(C_\varphi|_{H^2})]^{2/p}$ (see Theorem 3.8; we later show (Theorem 5.4) that this inequality is an equality). Thus, the condition $|\varphi'(a)|^{p/2} > r_e(C_\varphi|_{H^2})$ of the Main Theorem says that $|\varphi'(a)|$ exceeds $r_e(C_\varphi|_{H^p})$. Now, whenever $|\varphi'(a)|$ exceeds the essential spectral radius of $C_\varphi : H^p \to H^p$, the number $\varphi'(a)$ lies in the unbounded component of the essential resolvent of C_φ. On the other hand, $\varphi'(a)$ is always in the *spectrum* of $C_\varphi : H^p \to H^p$ (the function $z - a$ is easily seen to lie outside the range of the operator: see Proposition 3.3 for a proof in the case $a = 0$); thus, Fredholm theory guarantees that $\varphi'(a)$ must be an eigenvalue of C_φ on H^p. Since the Koenigs eigenfunction σ for φ spans the eigenspace corresponding to $\varphi'(a)$, we must have $\sigma \in H^p$. The preceding outline glosses over one difficulty: Fredholm theory is generally developed in a Hilbert- or Banach-space context; however, the Hardy spaces H^p for $p < 1$ are not Banach spaces. Fortunately, customary Fredholm facts continue to hold for H^p when $p < 1$ ([27]; see also the following section).

The principal ancestors of this paper are works [28] and [25]. In [28], Shapiro, Smith, and Stegenga study the relationship between the geometry of the image of the Koenigs function of a univalent map φ and the compactness of the composition operator $C_\varphi : H^p \to H^p$ (compactness is independent of p). They relate noncompactness of all powers of C_φ to the existence of a sector or "twisted sector" in image $\sigma(U)$ of the Koenigs function of φ and establish that the existence of such a twisted sector shows σ fails to belong to H^p for some p. (Pietro Poggi-Corradini [18] recently showed that if $\sigma(U)$ contains no twisted sector, then σ must belong to H^p for all p.)

In [25], Shapiro shows that the essential norm $\mathcal{E}_\varphi$ of the composition operator C_φ on H^2 is given by

$$\mathcal{E}_\varphi = \limsup_{|w|\to 1^-} \left(\frac{N_\varphi(w)}{1-|w|}\right)^{1/2},$$

where N_φ denotes the Nevanlinna counting function of φ. Hence, by employing the (essential) spectral radius formula, we have

$$r_e(C_\varphi|_{H^2}) = \lim_{n\to\infty} (\mathcal{E}_{\varphi_n})^{1/n},$$

which allows us to restate our Main Theorem in purely function-theoretic terms.

Main Theorem. *If* $\varphi'(a)^{p/2} > \lim_{n\to\infty} (\mathcal{E}_{\varphi_n})^{1/n}$, *then* $\sigma \in H^p$.

For certain maps φ, the essential spectral radius $r_e(C_\varphi|_{H^2})$ may also be expressed in terms of angular derivatives at boundary fixed points. Consider the following simple situation. Suppose that φ is analytic on the closed disk fixing the point $a \in U$, that $\varphi(1) = 1$, and that $|\varphi(\zeta)| < 1$ for $\zeta \in \partial U \backslash \{1\}$. Then (see Theorem 4.1 or [6, Corollary 2.5])

$$r_e(C_\varphi|_{H^2}) = \left(\frac{1}{\varphi'(1)}\right)^{1/2}.$$

(We remark that $\varphi'(1) > 1$ by the Denjoy-Wolff Theorem.) Our work shows that for such a φ, the condition

$$|\varphi'(a)|^p > \frac{1}{\varphi'(1)} \tag{1.2}$$

is necessary and sufficient for σ to belong to H^p. For example, if $\varphi(z) = z^5/30 + z^4/30 + 14z/15$, then $a = 0$, and the preceding condition shows that the Koenigs eigenfunction σ for φ belongs to H^p precisely when $p < \log(37/30)/\log(15/14) \approx 3.04$.

By comparing the region $\sigma(U)$ of Figure 2 to a sector with angular opening approximately $\pi/3$, one may verify that our work yields Hardy-space membership for σ that is consistent with the figure. The verification requires Littlewood's Subordination Principle ([9, Theorem 1.7]), the observation that σ is univalent (because φ is), and the fact that a univalent mapping of U onto a sector with angular opening π/r, such as

$$S_r(z) := \left(\frac{1+z}{1-z}\right)^{1/r},$$

belongs to H^p precisely when $p < r$ (see the proof of Theorem 3.2 of [9]). Observe that the region $\sigma(U)$ in Figure 2 appears to be contained in a left-translate of the image of S_r for $r \approx 3$; thus by Littlewood's Principle, σ should be in H^p for $p \approx 3$. Moreover, because $\sigma(U)$ appears to contain the image $S_r(U)$ for $r \approx 3$, Littlewood's Principle also tells us that σ should not be contained in H^p for p significantly greater than 3.

This paper is organized as follows. In the next section, we discuss p-Banach spaces and the Fredholm theory of operators on such spaces. We also present a detailed description of Koenigs' solution to Schroeder's functional equation. In Section 3, we prove that the essential spectral radius of $C_\varphi : H^p \to H^p$, $r_e(C_\varphi|_{H^p})$, does not exceed $(r_e(C_\varphi|_{H^2}))^{2/p}$ and obtain as a corollary the Main Theorem. Section 4 contains our proof of the converse of the Main Theorem, given the additional hypothesis that φ be analytic on the closed disk. In Section 5, we show that for an arbitrary mapping $\psi : U \to U$ (which need *not* fix a point in U),

$$r_e(C_\psi|_{H^p}) = (r_e(C_\psi|_{H^2}))^{2/p},$$

which answers the obvious question about essential spectral radii raised by our work in Section 3. We also establish in Section 5 that spectral radius equals essential spectral radius for any Hardy-space composition operator induced by a mapping

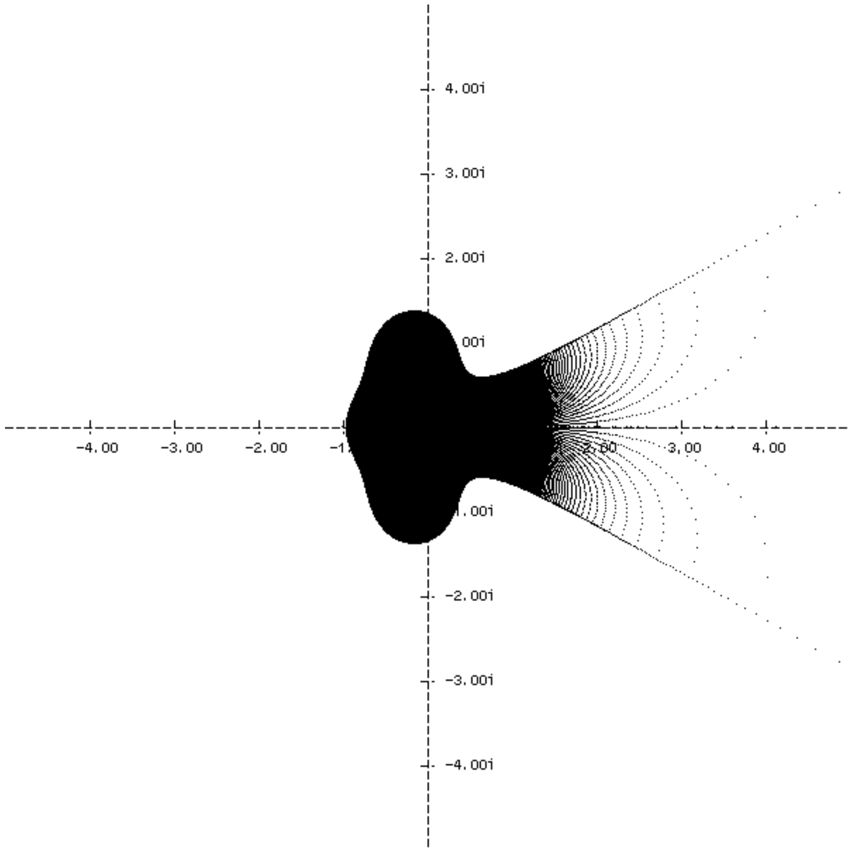

FIGURE 2. The shaded region is the image $\sigma(U)$ of the Koenigs eigenfunction for $\varphi(z) = z^5/30 + z^4/30 + 14z/15$ (the image was generated using formula (1.1) with $a = 0$).

with Denjoy-Wolff point on ∂U. In the final section, we discuss questions suggested by the results presented in this paper.

2. Preliminaries

In this section, we present some basic information about p-Banach spaces, Fredholm operators, and Koenigs eigenfunctions.

p-Banach spaces. We assume that readers are familiar with basic properties of Banach spaces. We must work in the more general context of p-Banach spaces in order to address the problem of Koenigs-function membership in H^p for $0 < p < 1$. Readers interested in our H^p-membership results only for the range $p \geq 1$ may wish to skip this section, for they will have no trouble adapting our results and arguments to the simpler Banach-space setting.

The definition of *p-Banach space* depends upon the notion of *p-norm.* Let $0 < p \leq 1$. We say that the function $\|\cdot\| : X \to [0, \infty)$ on the complex vector space X

is a *p-norm* provided that for $x, y \in X$ and $a \in \mathbf{C}$

(a) $\|x+y\| \le \|x\| + \|y\|$;
(b) $\|ax\| = |a|^p\|x\|$;
(c) $\|x\| = 0 \Rightarrow x = 0$.

Thus only property (b) distinguishes a p-norm from a norm (and a p-norm is a norm when $p = 1$). The reader may verify that for $0 < p \le 1$ the function defined on H^p by

$$f \mapsto \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p\, d\theta$$

is a p-norm. Here $f(e^{i\theta})$ denotes the radial limit of f at $e^{i\theta}$ (which exists at almost every point of ∂U) and the value of the integral on the right equals $\|f\|_p^p$ (see [9, Theorem 2.6]).

A p-norm on a complex vector space X induces an invariant metric d on X: $d(x, y) = \|x - y\|$. If X is complete in the metric d, we call X a p-Banach space. Note that p-Banach spaces are examples of locally bounded F spaces. Also note a 1-Banach space is a Banach space. The Hardy spaces H^p for $0 < p \le 1$ are p-Banach spaces (which may be viewed as closed subspaces of $L^p(\partial U)$; see [9, Theorem 3.3]).

As long as linear functionals do not enter the picture, many standard Banach-space facts carry over to p-Banach spaces, with similar proofs (see, e.g., [27]). Here's a listing of a few such facts.

(a) A linear map $T : X \to X$ is continuous if and only if it is bounded in the sense that
$$\|T\| := \sup\{\|Tx\| : x \in X, \|x\| \le 1\}$$
is finite.
(b) The collection of bounded linear operators on X, which we'll denote $L(X)$, is a p-Banach space with p-norm defined in (a).
(c) The collection of invertible elements in $L(X)$ is open and the spectrum of $T \in L(X)$ is contained in the closed disk centered at 0 of radius $\|T\|^{1/p}$.

To give the reader a sense of how Banach-space proofs are modified to yield p-Banach space facts, we'll sketch the proof of the second assertion of (c).

Proposition 2.1. *Suppose $|\lambda|^p > \|T\|$; then $T - \lambda I$ is invertible on the p-Banach space X.*

Proof. We have $T - \lambda I = -\lambda(I - (T/\lambda))$. Observe $\|T/\lambda\| = |1/\lambda|^p\|T\| < 1$. The reader may check that

$$\sum_{n=0}^{\infty} (T/\lambda)^n$$

represents an inverse of the operator $I - (T/\lambda) : X \to X$ and thus $T - \lambda I$ is invertible. $\square$

Fredholm operators. We present the definition of Fredholm operator in the context of p-Banach spaces. Throughout this subsection, X denotes a p-Banach space. Let Y and Z be subspaces of X. We say that X is the algebraic direct sum of Y and Z provided $X = Y + Z$ and $Y \cap Z = \emptyset$. If both Y and Z are closed we write $X = Y \oplus Z$ and remark that in this case the natural projections from X onto Y and from X onto Z are continuous.

We say that the bounded operator T on X is Fredholm provided that

(a) the range of T, ran T, is closed and has finite codimension in X, and
(b) the kernel of T, ker T, is finite dimensional and there is a closed subspace Y of X such that $X = \ker T \oplus Y$.

Note that if (a) holds, then $X = \text{ran } T \oplus M$ for some finite-dimensional subspace M of X (because finite-dimensional subspaces of X are closed).

We define the index i of the Fredholm operator T by

$$i(T) = \text{dimension}(\ker \ T) - \text{codimension}(\text{ran } T).$$

For our work, the crucial property of the index is its continuity as a mapping from the collection of all Fredholm operators on X into the integers, where the collection of Fredholm operators is topologized as a subset of $L(X)$ (see [27]). Observe that a Fredholm operator of index zero is either invertible (in $L(X)$) or has nontrivial kernel.

Fredholm operators may be characterized as "invertible operators modulo the compact operators". Recall that a bounded operator T is compact on X provided some open set containing 0 is mapped by T into a set that has compact closure; finite rank operators (ran T is finite dimensional) are examples of compact operators. The following result, which may be regarded as the fundamental theorem of Fredholm theory, explains what is meant by "invertible modulo the compact operators" (for a proof, see [27]).

Atkinson's Theorem. *Suppose that T is a bounded linear operator on the p-Banach space X. The following are equivalent.*

(a) *T is Fredholm;*
(b) *there is an operator $S \in L(X)$ such that both $I - ST$ and $I - TS$ are finite-rank;*
(c) *there is an operator $S \in L(X)$ such that both $I - ST$ and $I - TS$ are compact.*

Because Fredholm operators are "essentially invertible", we have the associated notions of "essential" spectrum, spectral radius, and norm. The *essential spectrum* of an operator T on the p-Banach space X is $\{\lambda \in \mathbf{C} : T - \lambda I \text{ is not Fredholm}\}$. The *essential spectral radius* of T, denoted $r_e(T)$, is, of course, the supremum of the moduli of elements in the essential spectrum of T. The *essential norm* of $T : X \to X$, denoted $\|T\|_e$, is given by

$$\|T\|_e = \inf\{\|T - K\| : K \text{ is compact on } X\}.$$

Observe that $\|T\|_e \le \|T\|$ and that by Atkinson's Theorem,

$$r_e(T)^p \le \|T\|_e.$$

We also have the essential spectral radius formula

$$r_e(T)^p = \lim_{n\to\infty} (\|T^n\|_e)^{1/n} \tag{2.1}$$

(see [11]).

The following observation is crucial to our work.

Proposition 2.2. *Suppose that $|\alpha| > r_e(T)$; then $T - \alpha I$ is Fredholm of index zero.*

Proof. That $T - \alpha I$ is Fredholm is obvious. To show its index is zero, we use the continuity of the Fredholm index. Choose $c > 1$ large enough so that $|c\alpha|^p > \|T\|$ and observe that $T - c\alpha I$ is Fredholm of index zero (in fact, $T - c\alpha I$ is invertible).

Now for each $s \in [1, c]$ the operator $T - s\alpha I$ is Fredholm (because α exceeds $r_e(T)$); thus,

$$g(s) := i(T - s\alpha I)$$

is a continuous mapping from $[1, c]$ into the integers. Because $g(c) = 0$, $g(1) = i(T - \alpha I) = 0$. □

Koenigs eigenfunctions. We call any function mapping the open unit disk U into itself a *self-map* of U. Let φ be a holomorphic self-map of U that is not an elliptic automorphism of U (elliptic automorphisms are those holomorphic automorphisms that fix a point in U). Orbits of points in U under φ all converge to a distinguished fixed point ω of φ called the *Denjoy-Wolff point* of φ.

Denjoy-Wolff Theorem. *There is a point ω in the closure of U such that for each $z \in U$,*

$$\varphi_n(z) \to \omega \qquad \textit{as} \qquad n \to \infty.$$

The Denjoy-Wolff point ω of φ may be characterized as follows:

- if $|\omega| < 1$, then $\varphi(\omega) = \omega$ and $|\varphi'(\omega)| < 1$;
- if $|\omega| = 1$, then $\varphi(\omega) = \omega$ and $0 < \varphi'(\omega) \le 1$,

where when $\omega \in \partial U$, $\varphi(\omega)$ is the angular limit of φ at ω and $\varphi'(\omega)$ is the angular derivative of φ at ω. Recall that φ is said to have angular derivative at $\zeta \in \partial U$ if there is an $\eta \in \partial U$ such that

$$\angle \lim_{z \to \zeta} \frac{\varphi(z) - \eta}{z - \zeta}$$

is finite, where $\angle \lim$ denotes the angular (or nontangential) limit. By the Julia-Carathéodory Theorem, φ has finite angular derivative at ζ if and only if φ' has angular limit at ζ while φ has angular limit of modulus 1 at ζ. For these results, the reader may consult [7, Chapter 2] or [26, Chapters 4 and 5].

If the Denjoy-Wolff point ω of φ lies on ∂U or if $\omega \in U$ and $\varphi'(\omega) \neq 0$, then φ has a *linear-fractional model*: there exist a nonconstant holomorphic function f and a linear-fractional transformation ψ such that

$$f \circ \varphi = \psi \circ f. \tag{2.2}$$

This model is the work of a number of authors, whose efforts stretch over nine decades: Koenigs ([15]: 1884), Valiron ([29]: 1931), Baker and Pommerenke ([21, 1]: 1979), and Cowen ([5]: 1981). When ω is in U, the linear-fractional map ψ in (2.2) may be taken to be a dilation, $\psi(z) = \lambda z$ for some scalar λ, and (2.2) becomes Schroeder's functional equation.

For the remainder of this paper we will assume that the Denjoy-Wolff point ω of φ lies in U and will focus on the mean-growth of solutions of Schroeder's functional equation. In the analyzing mean-growth (i.e., Hardy-space membership), we may assume without loss of generality that $\omega = 0$. The reason is that each solution to Schroeder's equation corresponding to a self-map φ having nonzero Denjoy-Wolff point in U may be transformed through composition with a disc automorphism into a solution corresponding to a self-map having Denjoy-Wolff point 0. The process is simple. Suppose $\omega \in U\backslash\{0\}$ is the Denjoy-Wolff point of φ and that f is a holomorphic solution to Schroeder's equation:

$$f \circ \varphi = \lambda f \quad \text{for some scalar } \lambda.$$

Set

$$\alpha_\omega(z) := \frac{\omega - z}{1 - \bar{\omega} z} \qquad \text{and} \qquad \Phi(z) := (\alpha_\omega \circ \varphi \circ \alpha_\omega)(z);$$

thus, Φ is a self-map of U fixing 0 that is conjugate to φ via the (self-inverse) automorphism α_ω. Because $f \circ \varphi = \lambda f$, the function $g := f \circ \alpha_\omega$ satisfies

$$g \circ \Phi = \lambda g. \tag{2.3}$$

Observe that g is a solution to Schroeder's equation for a self-map with Denjoy-Wolff point 0 and that information about the the mean growth of g transfers to $f = g \circ \alpha_\omega$ since composition with α_ω preserves H^p.

Let φ be a nonconstant holomorphic self-map of U such that $\varphi(0) = 0$ and φ is not an elliptic automorphism. Suppose that f is a holomorphic function on U satisfying Schroeder's functional equation

$$f \circ \varphi = \lambda f. \tag{2.4}$$

We record some simple observations.

- If $\lambda = 0$, then using analyticity and the fact that φ is nonconstant, we must have $f \equiv 0$.
- If $\lambda = 1$, then $f \circ \varphi_n = f$ for all positive integers n, and by the Denjoy-Wolff theorem f must be constant: $f(z) = f(0)$ for all $z \in U$.
- If $\lambda \neq 1$, then $f(0) = 0$.
- Suppose that $\lambda \notin \{0, 1\}$ and that $\varphi'(0) = 0$. Then $f \equiv 0$ (otherwise note that the order of the zero of $f \circ \varphi$ at 0 exceeds the order of the zero of f at zero, contradicting (2.4)).

Thus, to avoid trivial situations, we add the assumption that $\varphi'(0) \neq 0$ and seek nonconstant solutions of (2.4). Summarizing our assumptions, we have: φ is an analytic self-map of U that fixes 0 and satisfies $0 < |\varphi'(0)| < 1$ (the condition $|\varphi'(0)| < 1$ is equivalent—by the Schwarz Lemma—to our requirement that φ not be an elliptic automorphism).

Koenigs obtained solutions to Schroeder's equation (2.4) as limits of sequences of normalized iterates of φ. We motivate his approach as follows. Suppose that f satisfies (2.4) and that $f'(0) \neq 0$. Differentiating both sides of (2.4) and evaluating the result at zero, we find that $\lambda = \varphi'(0)$. Thus for each nonnegative integer n, we have

$$f \circ \varphi_n = \varphi'(0)^n f.$$

Differentiating both sides of the preceding equation and rearranging the result yields

$$f'(\varphi_n)\frac{\varphi_n'}{\varphi'(0)^n} = f'.$$

Because φ_n approaches zero uniformly on compact subsets of U (by, e.g., the Schwarz Lemma), the sequence

$$f'(0)\frac{\varphi_n'}{\varphi'(0)^n} \to f' \tag{2.5}$$

uniformly on compact subsets of U. Now, integration of both sides of (2.5) over the line segment with endpoints 0 and z ($z \in U$) yields

$$f(z) = f'(0) \lim_{n\to\infty} \frac{\varphi_n(z)}{\varphi'(0)^n}.$$

Hence, if f is a solution to Schroeder's equation such that $f'(0) \neq 0$, then f must be a constant times the pointwise limit of the sequence

$$\frac{\varphi_n}{\varphi'(0)^n} \tag{2.6}$$

of normalized iterates of φ.

Koenigs showed that the sequence (2.6) converges uniformly on compact subsets of U and that its limit, which we label σ, satisfies

$$\sigma \circ \varphi = \varphi'(0)\sigma \qquad \text{and} \qquad \sigma'(0) = 1. \tag{2.7}$$

That

$$\sigma = \lim_{n\to\infty} \frac{\varphi_n}{\varphi'(0)^n} \tag{2.8}$$

satisfies (2.7) is easy to establish; Koenigs also showed that σ is the unique function satisfying (2.7) (uniqueness also follows from the argument of the preceding paragraph). We call (2.6) the *Koenigs sequence* of φ, and its limit σ, the *Koenigs eigenfunction* of φ. Any eigenfunction corresponding to the eigenvalue $\varphi'(0)$ must be a constant multiple of the Koenigs eigenfunction σ.

Now, observe that for any nonnegative integer n, the function σ^n satisfies

$$\sigma^n \circ \varphi = \varphi'(0)^n \sigma^n.$$

Thus the set of eigenvalues of C_φ, viewed as a linear transformation of the vector space $H(U)$ of all functions holomorphic on U, contains

$$E := \{\varphi'(0)^n : n = 0, 1, 2, \dots\}.$$

Koenigs' work shows that E is precisely the set of eigenvalues of $C_\varphi : H(U) \to H(U)$ and that each eigenvalue is simple; in other words, nonzero solutions to Schroeder's equation (2.4) exist only when $\lambda = \varphi'(0)^n$ for some nonnegative integer n, in which case $f = c\sigma^n$ for some scalar c. Thus information about the growth of σ yields information about the growth of *every* solution of Schroeder's equation for φ.

Notational convention. *For the remainder of this paper we assume that φ is a holomorphic self-map of U such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$ and that σ is defined by* (2.8).

By the Schwarz Lemma, our assumption that $|\varphi'(0)| < 1$ simply insures that φ does not map U one-to-one onto itself (since φ fixes the origin, this is equivalent to saying that φ is not a rotation).

Because σ is the limit of a sequence of normalized iterates of φ, in most cases obtaining a closed-form formula for σ is impossible. Thus one shouldn't in general expect sharp growth estimates for σ to come easily. There are, however, the following elementary results.

Proposition 2.3. *The Koenigs function σ for a univalent self-map φ must belong to H^p for all $p < 1/2$.*

Proof. By (2.8) and Hurwitz's Theorem, the Koenigs function for a univalent φ must itself be univalent. Since any univalent function belongs to H^p for all $p < 1/2$ (see, e.g., [9, Theorem 3.16]), the proposition follows. □

The following result is well-known (for a proof, see [4] or [26, Section 6.4]).

Proposition 2.4. *Suppose that for some positive integer k, the composition operator $C_{\varphi_k} : H^2 \to H^2$ is compact; then the Koenigs function σ of φ belongs to H^p for all p.*

Compactness of C_φ on H^p spaces has been characterized in terms of the distribution of values of φ (see [25] or [26]). If φ is univalent this criterion reduces to *non-existence of the angular derivative at every point of the unit circle.* The connection between compactness and Koenigs eigenfunctions is studied further in [28] (see also [26, Chapter 9]).

3. A sufficient condition for inclusion in H^p

Recall that φ denotes a holomorphic self-map of the open unit disk U such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, while σ denotes the Koenigs eigenfunction corresponding to φ, so that

$$\sigma \circ \varphi = \varphi'(0)\sigma \qquad \text{and} \qquad \sigma'(0) = 1.$$

In this section we present for $0 < p < \infty$ a sufficient condition for σ to belong to H^p; in the next section we show that the condition is necessary when φ extends analytically across ∂U.

Koenigs eigenfunctions need not belong to any H^p classes; in fact, if φ is inner then its Koenigs function fails to belong to the Nevanlinna class. Recall that the Nevanlinna class $\mathcal{N}$ contains all of the Hardy spaces and consists of those functions f holomorphic on U such that

$$\sup\{\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta : r \in [0, 1)\} < \infty.$$

Recall also that each function f in $\mathcal{N}$ has finite radial limit at almost every point of ∂U and that if $f^* : \partial U \to \mathbf{C}$ denotes the radial limit function of f, then

$$\log |f^*| \in L^1(\partial U)$$

(see, e.g. [9, Theorem 2.2]).

The following Lemma is well-known; see, for example, [23, Theorem 2] or [7, Proposition 2.25].

Lemma 3.1. *Suppose that φ is an inner function and that g is an analytic function on U belonging to the Nevanlinna class. Then*

$$(g \circ \varphi)^* = g^* \circ \varphi^*$$

almost everywhere on ∂U.

Sketch of proof. Use Lindelöf's theorem (see [22, p. 259]) to establish the result when g is bounded. Then use the fact than every function in $\mathcal{N}$ is a quotient of bounded functions (see [9, Theorem 2.1]). □

Let m denote normalized Lebesgue measure on ∂U.

Proposition 3.2. *Suppose that φ is an inner function such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$. Then the Koenigs map σ for φ doesn't belong to the Nevanlinna class.*

Proof. Suppose that σ does belong to the Nevanlinna class. Using integrability of $\log|\sigma^*|$ and the fact that φ^* is a measure-preserving transformation on ∂U (see [17]), and then applying the preceding lemma, we have

$$\begin{aligned}\int_{\partial U} \log|\sigma^*|dm &= \int_0^{2\pi} \log|\sigma^* \circ \varphi^*|dm \\ &= \int_{\partial U} \log|\varphi'(0)\sigma^*|dm \\ &= \int_{\partial U} \log|\sigma^*|dm + \log|\varphi'(0)|.\end{aligned}$$

Thus $\log|\varphi'(0)| = 0$, which contradicts our hypothesis that $0 < |\varphi'(0)| < 1$. □

We remark that in the preceding proposition the assumption that φ be inner cannot be weakened to the assumption that φ^* have modulus 1 on a set of positive measure. Example 1.6 from [28] dramatizes the difference between these hypotheses by showing there exists a self map φ with radial limits of modulus one on a set of positive measure whose Koenigs eigenfunction belongs to H^p for all p.

We now seek a sufficient condition for the Koenigs function σ of φ to belong to H^p. Because

$$f \circ \varphi = \varphi'(0)f$$

if and only if f is a constant multiple of σ, we see that $\varphi'(0)$ is an eigenvalue of $C_\varphi : H^p \to H^p$ if and only if σ belongs to H^p. Thus information about the point spectrum of the composition operator $C_\varphi : H^p \to H^p$ yields information about the Hardy classes to which the Koenigs function of φ belongs. Although $\varphi'(0)$ may fail to be in the point spectrum of $C_\varphi : H^p \to H^p$ (when, e.g., φ is inner), it is always in the spectrum. In fact, $\varphi'(0)^n$ belongs to the spectrum for every nonnegative integer n. This may be seen for C_φ operating on the Hilbert space H^2 through a matrix computation (see, e.g., [7, Proposition 7.32]). For the H^p setting we employ a different argument.

Proposition 3.3. *For each $p > 0$, the spectrum of $C_\varphi : H^p \to H^p$ contains $\varphi'(0)^n$ for every nonnegative integer n.*

Proof. Let $p > 0$ and let n be a nonnegative integer. If $n = 0$, then $\varphi'(0)^n = 1$ is an eigenvalue for C_φ. Suppose that $n = 1$ and that the range $C_\varphi - \varphi'(0)I$ contains the H^p function $g(z) = z$. Then, letting f denote a pre-image of g, we have

$$f(\varphi(z)) - \varphi'(0)f(z) = z$$

for each z. Differentiate both sides of the preceding equation and let $z = 0$ to obtain the contradiction $0 = 1$. Thus $C_\varphi - \varphi'(0)I$ is not onto so that $\varphi'(0)$ is in the spectrum of C_φ.

For $n > 1$, an inductive argument (or one based on power series) shows that the range of the operator $C_\varphi - \varphi'(0)^n I$ cannot contain the H^p function $g(z) = z^n$. Thus $\varphi'(0)^n$ is contained in the spectrum of C_φ for every nonnegative integer n. □

We will obtain information about the point spectrum of C_φ using the following well-known fact from Fredholm theory.

Proposition 3.4. *Suppose that T is a p-Banach-space operator and λ is a point in the spectrum of T such that $T-\lambda I$ is Fredholm of index 0. Then λ is an eigenvalue of T.*

Proof. Being Fredholm of index 0, the operator $T-\lambda I$ has closed range of codimension equal to the dimension of $\ker(T-\lambda)$. Thus $T-\lambda$ is non-invertible if and only if λ is an eigenvalue. □

The preceding two propositions show that $\varphi'(0)$ is an eigenvalue of $C_\varphi : H^p \to H^p$ provided $(C_\varphi - \varphi'(0)I) : H^p \to H^p$ is Fredholm of index zero. By Proposition 2.2, the operator $(C_\varphi - \varphi'(0)I) : H^p \to H^p$ is Fredholm of index zero whenever $|\varphi'(0)|$ exceeds the essential spectral radius of C_φ on H^p, $r_e(C_\varphi|_{H^p})$; thus, we have

σ belongs to H^p whenever $|\varphi'(0)| > r_e(C_\varphi|_{H^p})$.

Thus we seek to determine the essential spectral radius of C_φ acting on H^p.

The essential spectral radius of C_φ acting on the Hilbert space H^2 has been calculated. In [25], Shapiro proves that the essential norm $\mathcal{E}_\varphi$ of the composition operator C_φ on H^2 is given by

$$\mathcal{E}_\varphi = \limsup_{|w|\to 1^-} \left(\frac{N_\varphi(w)}{1-|w|}\right)^{1/2},$$

where N_φ is the Nevanlinna counting function of φ. For a holomorphic map φ taking U into itself N_φ is defined on $U\backslash\{\varphi(0)\}$ by

$$N_\varphi(w) = \sum_{\varphi(z)=w} \log(1/|z|), \tag{3.1}$$

where multiplicities are counted and $N_\varphi(w)$ is taken to be zero if w is not in the range of φ. Via the (essential) spectral radius formula, Shapiro's result identifies the essential spectral radius for a composition operator on H^2.

We will show that

$$r_e(C_\varphi|_{H^p}) = \left(r_e(C_\varphi|_{H^2})\right)^{2/p}.$$

Because only the inequality

$$\left(r_e(C_\varphi|_{H^2})\right)^{2/p} \geq r_e(C_\varphi|_{H^p}) \tag{3.2}$$

is needed to obtain our Main Theorem, we postpone the proof of the reverse inequality until Section 5 (see Theorem 5.4). Our proof of (3.2) requires three lemmas. We will continue to assume that $\varphi(0)=0$, but the proof of inequality (3.2) will not require the additional assumption that $0<|\varphi'(0)|<1$.

Lemma 3.5. *Suppose that $T : X \to X$ is a continuous linear operator on the p-Banach space X and that Y is a closed, finite codimensional subspace X that is invariant under T. If $T : Y \to Y$ is invertible, then T is Fredholm on X.*

Proof. Suppose that $T : Y \to Y$ is invertible. Because Y is closed and has finite codimension in X,

$$X = Y \oplus M$$

for some finite-dimensional subspace $M \subset X$. Define $S : X \to X$ by $S = T^{-1}$ on Y and $S \equiv 0$ on M. Letting P_M denote the projection of X onto M, we have

$$I - TS = P_M \qquad \text{and} \qquad I - ST = (I - ST)P_M.$$

Thus both $I - TS$ and $I - ST$ are finite rank operators and thus by Atkinson's Theorem T is Fredholm. □

Our next two lemmas involve the operator norm of C_φ acting on H^p. Because the notions of norm differ for the $0 < p < 1$ and the $p \geq 1$ cases, we introduce below the "norm" $\|\cdot\|_*$, which will allow us to treat simultaneously the $0 < p < 1$ and $p \geq 1$ cases. Let $(X)_1$ represent the open unit ball of the space X so that, for example,

$$(H^p)_1 = \{f \in H^p : \int_{\partial U} |f|^p dm < 1\}.$$

For each closed subspace M of H^p that is invariant under the operator $T : H^p \to H^p$, we define

$$\|T|_M\|_* = \sup\{\left(\int_{\partial U} |Tf|^p dm\right)^{1/p} : f \in (M)_1\}.$$

Thus for $1 \leq p < \infty$, $\|C_\varphi|_M\|_*$ is simply the operator norm of C_φ on the Banach space M, while for $0 < p < 1$, $\|C_\varphi|_M\|_*^p$ is the norm of C_φ on the p-Banach space M.

Observe that for any nonnegative integer m the subspace $z^m H^p$ is closed in H^p and (by our standing assumption that $\varphi(0) = 0$) invariant under C_φ .

Lemma 3.6. *Let $0 < p < \infty$ and let φ be an analytic self-map of U fixing 0. Then*

$$r_e(C_\varphi|_{H^p}) \leq \lim_{m\to\infty} \|C_\varphi|_{z^m H^p}\|_* \ . \tag{3.3}$$

Proof. Because the sequence of norms on the right-hand side of (3.3) is decreasing, the limit in (3.3) exists. Let m be a nonnegative integer and suppose

$$|\lambda| > \|C_\varphi|_{z^m H^p}\|_* \ . \tag{3.4}$$

For $p \geq 1$, the right-hand side of (3.4) is simply the operator norm of C_φ restricted to $z^m H^p$, so $C_\varphi - \lambda I$ is invertible on $z^m H^p$. For $0 < p < 1$, (3.4) implies λ^p exceeds the norm of C_φ on the p-Banach space $z^m H^p$ and thus, once again, $C_\varphi - \lambda I$ is invertible on $z^m H^p$ (see Proposition 2.1). Because $z^m H^p$ has finite codimension in H^p, Lemma 3.5 yields $C_\varphi - \lambda I$ is Fredholm, completing the proof. □

Lemma 3.7. *Suppose that φ is an analytic self-map of U fixing 0 and that $0 < p < \infty$. Then*

$$\lim_{m\to\infty} \|C_\varphi|_{z^m H^p}\|_* = \left(\lim_{j\to\infty} \|C_\varphi|_{z^j H^2}\|_*\right)^{2/p} . \tag{3.5}$$

Proof. Recall that $(X)_1$ represents the unit ball of the space X. Choose the positive integer N large enough so that $Np \geq 2$; let k be an arbitrary positive integer. Suppose $f \in (z^{Nk} H^p)_1$ so that $f = z^{Nk} g$ for some $g \in (H^p)_1$. Let B be the

Blaschke factor of g. We have

$$\begin{aligned}\|C_\varphi f\|_{H^p} &= (\int_{\partial U} |\varphi^{Nk}|^p |g\circ\varphi|^p dm)^{1/p}\\ &= (\int_{\partial U} |\varphi^{Nk}|^p |B\circ\varphi|^p |(g/B)\circ\varphi|^p dm)^{1/p}\\ &\le (\int_{\partial U} |\varphi^k|^2 |(g/B)^{p/2}\circ\varphi|^2 dm)^{1/p}\\ &= \left(\|C_\varphi z^k (g/B)^{p/2}\|_{H^2}\right)^{2/p}\\ &\le (\|C_\varphi|_{z^kH^2}\|_*)^{2/p},\end{aligned}$$

where we have used $Np \ge 2$ and $|\varphi| \le 1$ as well as $|B\circ\varphi| \le 1$ to obtain the first inequality appearing in the display above. The last inequality follows because $z^k(g/B)^{p/2}$ belongs to $(H^2)_1$. Because $f \in (z^{Nk}H^p)_1$ is arbitrary, we may conclude that

$$\|C_\varphi|_{z^{Nk}H^p}\|_* \le (\|C_\varphi|_{z^kH^2}\|_*)^{2/p}. \tag{3.6}$$

Let $k \to \infty$ and use the fact that the norm sequences on both sides of (3.5) are decreasing to obtain

$$\lim_{m\to\infty} \|C_\varphi|_{z^mH^p}\|_* \le \left(\lim_{j\to\infty} \|C_\varphi|_{z^jH^2}\|_*\right)^{2/p}.$$

Now let N be a fixed integer larger than p. Let k be an arbitrary positive integer. Suppose that $f \in (z^{Nk}H^2)_1$ so that $f = z^{Nk}g$ for some $g \in (H^2)_1$. Let B be the Blaschke factor of g. We have

$$\begin{aligned}\|C_\varphi f\|_{H^2} &= (\int_{\partial U} |\varphi^{Nk}|^2 |g\circ\varphi|^2 dm)^{1/2}\\ &= (\int_{\partial U} |\varphi^{2k}|^N |B\circ\varphi|^2 |(g/B)\circ\varphi|^2 dm)^{1/2}\\ &\le (\int_{\partial U} |\varphi^{2k}|^p |(g/B)^{2/p}\circ\varphi|^p dm)^{1/2}\\ &\le \left(\|C_\varphi z^{2k} (g/B)^{2/p}\|_{H^p}\right)^{p/2}\\ &\le (\|C_\varphi|_{z^{2k}H^p}\|_*)^{p/2}.\end{aligned}$$

Because $f \in (z^{Nk}H^2)_1$ is arbitrary, we have

$$\|C_\varphi|_{z^{Nk}H^2}\|_* \le (\|C_\varphi|_{z^{2k}H^p}\|_*)^{p/2}.$$

Since k is arbitrary, we have

$$\lim_{m\to\infty} \|C_\varphi|_{z^mH^p}\|_* \ge \left(\lim_{k\to\infty} \|C_\varphi|_{z^kH^2}\|_*\right)^{2/p}$$

which completes the proof of the lemma. □

In [25], Shapiro proves that

$$\|C_\varphi|_{H^2}\|_e = \lim_{m\to\infty} \|C_\varphi|_{z^mH^2}\|_*. \tag{3.7}$$

(see [25, Section 5.3, equation (2)]), where $\|\cdot\|_e$ represents essential norm. Shapiro's result and the lemmas above yield the following.

Theorem 3.8. *Suppose φ is an analytic self-map of U fixing 0. Then*

$$r_e(C_\varphi|_{H^p}) \le \left(r_e(C_\varphi|_{H^2})\right)^{2/p}.$$

Proof. Using Lemmas 3.6 and 3.7, and equation (3.7), we have

$$\begin{aligned} r_e(C_\varphi|_{H^p}) &\le \lim_{m\to\infty} \|C_\varphi|_{z^m H^p}\|_* \\ &= \left(\|C_\varphi|_{H^2}\|_e\right)^{2/p}. \end{aligned}$$

Replacing φ with φ_n and taking n-th roots yields

$$[r_e(C_{\varphi_n}|_{H^p})]^{1/n} \le \left(\|C_{\varphi_n}|_{H^2}\|_e^{1/n}\right)^{2/p}.$$

Now, to obtain the theorem, let $n \to \infty$, and use the (essential) spectral-radius formula along with the equality $r_e(T^n) = r_e(T)^n$ (which is valid—with the usual proof—for any p-Banach space operator T). □

Main Theorem. *Let $0 < p < \infty$ and let φ be an analytic self-map of U such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$. Suppose that*

$$|\varphi'(0)|^{p/2} > r_e(C_\varphi|_{H^2}); \tag{3.8}$$

then the Koenigs function σ for φ belongs to H^p.

Proof. By Theorem 3.8, $|\varphi'(0)|$ exceeds the essential spectral radius of $C_\varphi : H^p \to H^p$. Hence, $(C_\varphi - \varphi'(0)I) : H^p \to H^p$ must be Fredholm of index 0 (see Proposition 2.2). However $\varphi'(0)$ is in the spectrum of C_φ and thus must be an eigenvalue for $C_\varphi : H^p \to H^p$. Since only constant multiples of σ may serve as corresponding eigenfunctions, we must have $\sigma \in H^p$. □

Remarks. (1) In the next section we prove that the sufficient condition (3.8) for σ to belong to H^p is *necessary* at least when φ extends analytically across the boundary.

(2) Our Main Theorem may be viewed as a generalization of the Proposition 2.4 presented in the preceding section. The point here is that the quantity on the right-hand side of (3.8) is zero whenever some power of C_φ is compact. An operator T whose essential spectral radius is 0 is called a *Riesz operator*. The existence of Riesz composition operators that are not power compact is established in [2]. A characterization of Riesz composition operators with univalent symbol is contained in [19].

For φ having bounded valence, we may combine the Main Theorem with Theorem 3.5 of [25] to obtain the following corollary; in the statement $\varphi'(\zeta)$ represents the angular derivative of φ at ζ and should be interpreted as ∞ when this angular derivative doesn't exist.

Corollary 3.9. *Suppose that φ is a self-map having finite valence M. If*

$$|\varphi'(0)| > M \sup\{|\varphi'(\zeta)|^{-1/p} : \zeta \in \partial U\},$$

then the Koenigs function σ for φ belongs to H^p.

Proof. The quantity on the right-hand side of the inequality is greater than or equal to $\|C_\varphi|_{H^2}\|_e^{2/p}$ ([25, Theorem 3.5]), and, because the essential norm is greater than or equal to the essential spectral radius, the Main Theorem yields the corollary. □

When φ is analytic on the closure of U, the essential spectral radius $r_e(C_\varphi|_{H^2})$ may be calculated by evaluating the derivative of an appropriate iterate of φ at points on ∂U that are fixed for that iterate (see Theorem 4.1). This calculation is especially simple when the only points in $S := \varphi(\partial U) \cap \partial U$ are fixed by φ—in this case the "appropriate iterate" is the first one, and the Main Theorem (along with Theorem 4.1) yields the following:

$$(3.9) \qquad |\varphi'(0)|^{p/2} > \max\{\varphi'(\zeta)^{-1/2} : \zeta \in S\} \qquad \text{implies} \qquad \sigma \in H^p.$$

That the quantity $\max\{\varphi'(\zeta)^{-1/2} : \zeta \in S\}$ in (3.9) is $r_e(C_\varphi|_{H^2})$ also follows from [6, Corollary 2.5].

Example. Suppose $\varphi(z) = z/(2 - z^4)$. Then $S = \{1, i, -1, -i\}$ contains only boundary points fixed by φ, so that (3.9) applies and shows that $\sigma \in H^p$ for each $p < \log(5)/\log(2) \approx 2.322$ (the image of σ is Figure 1 on the first page of this paper). Moreover, Theorem 4.7 of the following section shows that this result is sharp: σ does *not* belong to H^p for any $p \geq \log(5)/\log(2)$. We note that this information about the H^p classes to which σ belongs determines the point spectrum of C_φ on H^p. For example, the point spectrum of $C_\varphi : H^1 \to H^1$ equals $\{1, \varphi'(0), \varphi'(0)^2\}$.

4. A necessary condition for inclusion in H^p

We continue to assume that φ is a holomorphic self-map of U such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$ and that σ is the Koenigs eigenfunction for φ. In this section, we show that the sufficient condition of the Main Theorem for σ to belong to H^p is *necessary* when φ is analytic on the closed disk. Our work depends upon the formula for essential spectral radius contained in part (b) of Theorem 4.1 below. Part (a) of the theorem is due to Kamowitz ([14, Theorem A (part 3)]), while part (b) is stated on page 296 of [7].

Theorem 4.1. *Suppose that $\varphi : U \to U$ is analytic on the closure of U and that φ is not an inner function. Then*

(a) *there is a positive integer N such that the set*

$$S_N := \{\varphi_N(\omega) : \omega \in \partial U, |\varphi_N(\omega)| = 1\}$$

is finite and consists only of fixed points of φ_N;

(b) $r_e(C_\varphi|_{H^2}) = \max\{\varphi_N'(\zeta)^{-1/(2N)} : \zeta \in S_N\}$.

Note that if S_N is empty, then $\|\varphi_N\|_\infty < 1$ so that C_{φ_N} is compact, and thus, $r_e(C_\varphi|_{H^2}) = 0$. Hence, $\max\{\varphi_N'(\zeta)^{-1/(2N)} : \zeta \in S_N\}$ should be interpreted as 0 when S_N is empty.

We've already noted that part (a) of Theorem 4.1 is proved in [14]. The maximum of part (b) also plays a role in [14], and one may prove part (b) by modifying arguments and results contained in that work. In particular, the proof of [14, Theorem 3.8] may be modified to show that $r_e(C_\varphi|_{H^2}) \leq \max\{\varphi_N'(\zeta)^{-1/(2N)} : \zeta \in S_N\}$. We prefer to present here a self-contained proof of part (b). The heart of our proof is contained in Lemmas 4.2, 4.3, and 4.4 below. In stating these lemmas we use the following notation:

- $E(\varphi) := \{\zeta \in \partial U : |\varphi(\zeta)| = 1\}$;
- $\delta(\varphi) := \max_{\zeta \in E(\varphi)} |\varphi'(\zeta)|^{-1}$;
- $\|C_\varphi|_{H^2}\|_e$ is the essential norm of $C_\varphi : H^2 \to H^2$.

If φ is analytic on the closure of U and is not inner, then the set $E(\varphi)$ is finite ([14, Lemma 1.3]), and we denote the number of elements in $E(\varphi)$ by $\#E(\varphi)$.

Lemma 4.2. *Suppose that $\varphi : U \to U$ is analytic on the closure of U and that φ is not an inner function. Then*

$$\delta(\varphi) \leq \|C_\varphi|_{H^2}\|_e^2 \leq (\#E(\varphi))\delta(\varphi).$$

Proof. By Theorem 3.3 of [25], the lower bound holds for *any* holomorphic self-map of U, with the boundary derivatives interpreted as angular derivatives.

To obtain the upper bound, we use Theorem 2.3 of [25]:

$$\|C_\varphi|_{H^2}\|_e^2 = \limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{1-|w|} .$$

For our special situation here, let $E = E(\varphi)$. Now φ is continuous on $\overline{U}$, $\varphi(\overline{U})$ contacts ∂U only at the points of $\varphi(E)$, and $N_\varphi(w)$ vanishes when w is outside $\varphi(U)$. Thus in our case the formula above simplifies to:

$$\|C_\varphi|_{H^2}\|_e^2 = \limsup_{w\to\varphi(E)} \frac{N_\varphi(w)}{1-|w|} = \max_{\eta\in\varphi(E)} \limsup_{w\to\eta} \frac{N_\varphi(w)}{1-|w|} .$$

Fix a point $\eta \in \varphi(E)$ that achieves the maximum on the right-hand side of the last display (the existence of this maximum is not in question since E, and therefore $\varphi(E)$, is finite). Let $\{\zeta_1, \zeta_2, \dots, \zeta_n\} = \varphi^{-1}(\{\eta\})$. All these pre-image points belong to E, of course, and since φ' vanishes at no point of E (by the Schwarz Lemma, $|\varphi'| \geq 1$ at each point of E) we know that φ is univalent in a neighborhood of each one. Thus (after a little arranging) we can find an open disk Δ_η centered at η whose inverse-image under φ consists of n disjoint open sets $V_j, 1 \leq j \leq n$, where for each j:

- V_j is a neighborhood of ζ_j,
- φ is one-to-one on V_j, and
- $\varphi(V_j) = \Delta_\eta$.

For each $w \in \Delta_\eta$ and $1 \leq j \leq n$ let $z_j(w)$ be the unique pre-image of w in V_j. Then we have, using our choice of η, and the fact that $w \to \eta \Leftrightarrow z_j(w) \to \zeta_j$ for each j:

$$\begin{aligned}
\|C_\varphi|_{H^2}\|_e^2 &= \limsup_{w\to\eta} \frac{N_\varphi(w)}{1-|w|} \\
&= \limsup_{w\to\eta} \sum_{j=1}^{n} \frac{\log(1/|z_j(w)|)}{1-|w|} \\
&\leq \sum_{j=1}^{n} \limsup_{w\to\eta} \frac{1-|z_j(w)|}{1-|w|} \\
&= \sum_{j=1}^{n} \limsup_{z\to\zeta_j} \frac{1-|z|}{1-|\varphi(z)|} \\
&= \sum_{j=1}^{n} \frac{1}{|\varphi'(\zeta_j)|} \quad \text{(by Julia-Carathéodory)} \\
&\leq (\#E)\delta(\varphi) .
\end{aligned}$$

□

Lemma 4.3. *Suppose that $\varphi : U \to U$ is analytic on the closure of U and that φ is not an inner function. Then*

$$r_e(C_\varphi|_{H^2})^2 = \lim_{n\to\infty} \delta(\varphi_n)^{1/n}.$$

Proof. The point here is that $E(\varphi_n) \subset E(\varphi)$ for every positive integer n, so by the preceding lemma,

$$\delta(\varphi_n) \le \|C_{\varphi_n}|_{H^2}\|_e^2 \le (\#E)\delta(\varphi_n),$$

where $E = E(\varphi)$ does not depend on n.

Now take n-th roots in the above string of inequalities and pass to the limit:

$$\limsup_{n\to\infty} \delta(\varphi_n)^{1/n} \le r_e(C_\varphi|_{H^2})^2 \le \liminf_{n\to\infty} \delta(\varphi_n)^{1/n},$$

which shows that $\delta(\varphi_n)^{1/n} \to r_e(C_\varphi|_{H^2})^2$ as $n \to \infty$. □

Lemma 4.4. *Suppose that $\varphi : U \to U$ is analytic on the closure of U, that φ is not an inner function, and that $\varphi(E(\varphi))$ consists entirely of fixed points of φ. Then*

$$r_e(C_\varphi|_{H^2})^2 = \max_{\eta\in\varphi(E)} \frac{1}{\varphi'(\eta)}\ .$$

Proof. For $\zeta \in E$ we have from the chain rule and the fact that $\varphi(\zeta)$ is a fixed point of φ:

$$\varphi_n'(\zeta) = \varphi_{n-1}'(\varphi(\zeta))\varphi'(\zeta) = \varphi'(\varphi(\zeta))^{n-1}\varphi'(\zeta).$$

Upon using this in the definition of $\delta(\varphi_n)$ we obtain:

$$\begin{aligned}
\delta(\varphi_n)^{1/n} &= \max_{\zeta\in E} \frac{1}{|\varphi'(\varphi(\zeta))|^{1-1/n}} \cdot \frac{1}{|\varphi'(\zeta)|^{1/n}} \\
&\to \max_{\zeta\in E} \frac{1}{|\varphi'(\varphi(\zeta))|} \quad \text{as } n \to \infty \\
&= \max_{\eta\in\varphi(E)} \frac{1}{|\varphi'(\eta)|},
\end{aligned}$$

so the desired result follows from Lemma 4.3 (note $|\varphi'(\eta)| = \varphi'(\eta)$ because η is a fixed point of φ). □

Proof of Theorem 4.1(b). Let N be the positive integer identified in part (a) of Theorem 4.1. To obtain part (b), apply the preceding lemma to φ_N instead of φ (with $E = E(\varphi_N)$), observe that $S_N = \varphi_N(E)$, and use the fact that $r_e(T^N) = r_e(T)^N$ for any bounded p-Banach space operator T. □

Remark. Theorem 4.1(b) should be compared with Corollary 2.5 of [6]: the boundary-fixed-point hypothesis of Theorem 4.1(b) is weaker but its smoothness hypothesis on φ is stronger.

The following corollary of Theorem 4.1 combined with our Main Theorem shows that the Koenigs eigenfunction of φ must belong to some H^p space whenever φ is analytic on the closure of U and not inner.

Corollary 4.5. *Suppose that $\varphi : U \to U$ is analytic on the closure of U and that φ is not an inner function. Then*

$$r_e(C_\varphi|_{H^2}) < 1.$$

Proof. Apply Theorem 4.1(b) and the Denjoy-Wolff Theorem. □

To obtain our necessary condition for Koenigs-map inclusion in H^p, we will require the following well-known fact about the growth of H^p functions.

Lemma 4.6. *Suppose that $f \in H^p$. Then*
$$|f(z)|(1-|z|)^{1/p} \to 0 \quad as \quad |z| \to 1^-.$$

Proof. Use the lemma on page 36 of [9], and then use density of the polynomials in H^p. □

We are now in a position to state and prove the promised necessary condition for Koenigs map inclusion in H^p. Our argument sharpens and generalizes the one used by Cowen to establish Theorem 3.8 of [6].

Theorem 4.7. *Suppose that φ is analytic on the closure of U and that for some $p \in (0,\infty)$ its Koenigs eigenfunction σ belongs to H^p. Then*
$$|\varphi'(0)|^{p/2} > r_e(C_\varphi|_{H^2}).$$

Proof. Each φ satisfying the hypotheses of this theorem satisfies the hypotheses of Theorem 4.1 ($\sigma \in H^p$ implies φ is not inner by Proposition 3.2). Thus there is a positive integer N such that the set $S_N := \{\varphi_N(\omega) : \omega \in \partial U, |\varphi_N(\omega)| = 1\}$ is finite and consists only of fixed points of φ_N.

Assume $N = 1$ so that $\varphi(\bar{U})$ contacts ∂U only at fixed points. Let ζ be one of those fixed points. The theorem for the $N = 1$ case follows if we can show
$$|\varphi'(0)|^{p/2} > \frac{1}{\sqrt{\varphi'(\zeta)}} \tag{4.1}$$
(by Theorem 4.1).

By the Denjoy-Wolff Theorem, $\varphi'(\zeta) > 1$ (recall again our standing assumption that $\varphi(0) = 0$). Thus there is a disk D centered at ζ such that φ has an inverse on D and $\varphi^{-1}(D) \subset D$. Define ψ on D by $\psi = \varphi^{-1}$ and set
$$\lambda = \psi'(\zeta) = 1/\varphi'(\zeta).$$
Koenigs' work, as discussed in the Introduction and in Section 2 (but now with D replacing U), shows that
$$\frac{\psi_n - \zeta}{\lambda^n}$$
converges uniformly as $n \to \infty$ on compact subsets of D to a univalent function $g : D \to \mathbf{C}$ such that
$$g \circ \psi = \lambda g.$$

Thus for any $z \in D$ we have $\psi_n(z) = g^{-1}(\lambda^n g(z))$. Because g^{-1} is conformal (at 0), there are uncountably many points z in $D \cap U$ such that the sequence $(\psi_n(z))$ approaches ζ nontangentially within $D \cap U$ (that is, $(\psi_n(z))$ lies within a triangle with vertex ζ that is contained in $D \cap U$). Choose such a point $z_0 \in D \cap U$ such that $\sigma(z_0)$ is nonzero. Set $z_k = \psi_k(z_0)$ and observe that because z_k approaches ζ nontangentially, there is a positive constant M such that
$$(1 - |z_k|) \geq M|\zeta - z_k|. \tag{4.2}$$
Using $\sigma \circ \varphi = \varphi'(0)\sigma$, we also have
$$\sigma(z_k) = [1/\varphi'(0)^k]\sigma(z_0). \tag{4.3}$$

Finally, observe that

$$\varphi'(\zeta)^k(\zeta - z_k) = \frac{\zeta - \psi_k(z_0)}{\lambda^k}$$

converges to $-g(z_0) \neq 0$ (g is univalent and $g(\zeta) = 0$). Therefore the sequence $\varphi'(\zeta)^k(\zeta - z_k)$ is bounded away from zero:

$$\varphi'(\zeta)^k|\zeta - z_k| \geq c > 0. \tag{4.4}$$

Putting (4.2), (4.3), and (4.4) together, we obtain

$$\begin{aligned} |\sigma(z_k)|(1-|z_k|)^{1/p} &\geq M|\sigma(z_k)||\zeta - z_k|^{1/p} \\ &= M|\sigma(z_0)|\frac{1}{\varphi'(\zeta)^{k/p}|\varphi'(0)|^k}[\varphi'(\zeta)^k|\zeta - z_k|]^{1/p} \\ &\geq Mc^{1/p}|\sigma(z_0)|\left(\frac{1}{\varphi'(\zeta)^{1/p}|\varphi'(0)|}\right)^k. \end{aligned}$$

Because $\sigma \in H^p$, the quantity $Mc^{1/p}|\sigma(z_0)|$ is positive, and k is arbitrary, Lemma 4.6 shows that we must have

$$|\varphi'(0)| > \varphi'(\zeta)^{-(1/p)},$$

which is equivalent to (4.1) so that the proof of the $N = 1$ case is complete.

Now suppose that $N > 1$ so that $\varphi_N(\bar{U})$ contacts the unit circle only at a finite number of fixed points of φ_N. Our work for the $N = 1$ case (replace φ with φ_N) shows that for any fixed point ζ of φ_N,

$$|\varphi_N'(0)| > [(\varphi_N)'(\zeta)]^{-(1/p)}.$$

Because $\varphi(0) = 0$, $\varphi_N'(0) = \varphi'(0)^N$ and thus we have

$$|\varphi'(0)| > [(\varphi_N)'(\zeta)]^{-1/(Np)}$$

or

$$|\varphi'(0)|^{p/2} > [(\varphi_N)'(\zeta)]^{-1/(2N)}.$$

Choosing ζ to yield the maximum on the right-hand side of (b) of Theorem 4.1, we have $|\varphi'(0)|^{p/2} > r_e(C_\varphi|_{H^2})$. $\square$

5. The essential spectral radius of a composition operator on H^p

In this section, we show that for an arbitrary analytic $\psi : U \to U$

$$r_e(C_\psi|_{H^p}) = (r_e(C_\psi|_{H^2}))^{2/p}. \tag{5.1}$$

We've already established that if $\psi(0) = 0$, then

$$r_e(C_\psi|_{H^p}) \leq (r_e(C_\psi|_{H^2}))^{2/p}$$

(see Theorem 3.8). Our first step in proving (5.1) is to show that the hypothesis $\psi(0) = 0$ is not needed to obtain the preceding inequality. We require the following two lemmas; in their statements, $r(C_\psi|_{H^p})$ denotes the spectral radius of $C_\psi : H^p \to H^p$.

Lemma 5.1. *Suppose that ψ is an arbitrary analytic self-map of U. Then*

$$r(C_\psi|_{H^p}) = r(C_\psi|_{H^2})^{2/p}.$$

Proof. Use the fact that $\|C_\psi|_{H^p}\|_* = \|C_\psi|_{H^2}\|_*^{2/p}$ (which follows from a simpler version of the argument yielding Lemma 3.7) and the spectral radius formula to obtain the result. □

Lemma 5.2. *Suppose that ψ is an analytic self-map of U with Denjoy-Wolff point $\omega \in \partial U$. Then*

$$r_e(C_\psi|_{H^2}) = \psi'(\omega)^{-1/2} = r(C_\psi|_{H^2}).$$

Proof. We have

$$r_e(C_\psi|_{H^2}) \le r(C_\psi|_{H^2}) = \psi'(\omega)^{-1/2},$$

where the equality above is part of Theorem 2.1 of [6]. To argue that

$$r_e(C_\psi|_{H^2}) \ge \psi'(\omega)^{-1/2},$$

we consider two cases: $\psi'(\omega) < 1$ (the hyperbolic case) and $\psi'(\omega) = 1$ (the parabolic case).

When $\psi'(\omega) < 1$, Theorem 4.5 of [6] shows that every point in the annulus $A := \{z : \psi'(\omega)^{1/2} < |z| < \psi'(\omega)^{-1/2}\}$ is an eigenvalue of C_ψ of infinite multiplicity so that the essential spectrum of C_ψ contains A. Thus, in the hyperbolic case, we have $r_e(C_\psi|_{H^2}) \ge \psi'(\omega)^{-1/2}$.

Suppose that $\psi'(\omega) = 1$, but $r_e < 1$. Then

$$C_\psi - I$$

is Fredholm of index zero and thus 1 must be an eigenvalue for C_ψ^* (because the constant function $g(z) = 1$ is in the kernel of $C_\psi - I$). Let f be a corresponding eigenfunction. We have for each positive integer n,

$$\begin{aligned} \frac{f^{(n)}(0)}{n!} &= \langle (C_\psi^*)^k f, z^n\rangle \\ &= \langle f, (\psi_k)^n\rangle \\ &\to \langle f, \omega^n\rangle \qquad (k \to \infty) \\ &= \bar{\omega}^n f(0). \end{aligned}$$

Thus all of the Taylor coefficients of f at 0 have modulus equal to $|f(0)|$; since f is in H^2, we must have $f \equiv 0$, a contradiction. Thus, in the parabolic case we have $r_e(C_\psi|_{H^2}) \ge 1 = \psi'(\omega)^{-1/2}$. □

Theorem 5.3. *Suppose that ψ is an arbitrary analytic self-map of U. Then*

$$r_e(C_\psi|_{H^p}) \le \left(r_e(C_\psi|_{H^2})\right)^{2/p}. \tag{5.2}$$

Proof. Let ω denote the Denjoy-Wolff point of ψ. If $|\omega| < 1$, then C_ψ is similar to a composition operator whose symbol fixes zero (see the discussion preceding equation (2.3)). Because similar operators have the same essential spectral radius, Theorem 3.8 gives us (5.2).

Suppose now that $|\omega| = 1$. We have

$$\begin{aligned} r_e(C_\psi|_{H^p}) &\le r(C_\psi|_{H^p}) \\ &= r(C_\psi|_{H^2})^{2/p} \qquad \text{(Lemma 5.1)} \\ &= r_e(C_\psi|_{H^2})^{2/p} \qquad \text{(Lemma 5.2)}, \end{aligned}$$

completing the proof. □

An alternative proof of the preceding theorem for $p \geq 1$ may be derived from the following essential-norm inequality, which holds for any analytic $\psi : U \to U$:

$$\|C_\psi|_{H^p}\|_e \leq 2^{|1-2/p|} \left(\|C_\psi|_{H^2}\|_e\right)^{2/p}. \tag{5.3}$$

For $p \geq 1$, Theorem 5.4 follows from the preceding inequality upon replacing ψ with ψ_n, taking n-th roots, letting $n \to \infty$, and then applying the essential spectral radius formula (2.1). We omit the proof of (5.3), except to note that it follows along the same lines as the proof of the corresponding part of the essential norm formula of [25], with the Littlewood-Paley formula (the case $p = 2$ of the formula below) for the H^2 norm replaced with the more general Hardy-Stein-Spencer identity ([13, Theorem 3.1, p. 67]):

$$\|f\|_p^p = \frac{p^2}{2} \int_U |f|^{p-2}|f'|^2 \log \frac{1}{|z|} d\nu(z) + |f(0)|^p \tag{5.4}$$

where $d\nu$ is normalized area measure on the unit disk, f is any function holomorphic on U, and $0 < p < \infty$.

While we omit the proof of inequality (5.3), the Hardy-Stein-Spencer identity is crucial to our next result, whose proof also uses ideas from [25].

Theorem 5.4. *Suppose ψ is an analytic self-map of U. Then*

$$r_e(C_\psi|_{H^p}) = \left(r_e(C_\psi|_{H^2})\right)^{2/p}.$$

Proof. In order to avoid separate treatment of the $p \geq 1$ and $p < 1$ cases, we define

$$\|C_\psi|_{H^p}\|_{e_*} = \inf\{\|(C_\psi - K)|_{H^p}\|_* : K \text{ is a compact operator on } H^p\}.$$

Thus $\|C_\psi|_{H^p}\|_{e_*}$ is the essential norm of C_ψ when $p \geq 1$, $\|C_\psi|_{H^p}\|_{e_*}^p$ is the essential norm of C_ψ when $0 < p < 1$.

By Theorem 5.3, we need only show that

$$r_e(C_\psi|_{H^p}) \geq \left(r_e(C_\psi|_{H^2})\right)^{2/p};$$

moreover, by the discussion preceding this theorem, to obtain this inequality it suffices to show that

$$\|C_\psi|_{H^p}\|_{e_*} \geq \left(\|C_\psi|_{H^2}\|_{e_*}\right)^{2/p}. \tag{5.5}$$

Our proof of the inequality (5.5) follows closely the argument presented in Section 5.4 of [25]. Fix $a \in U$ and $p \in (0, \infty)$. Let

$$\upsilon_a(z) = \frac{a - z}{1 - \bar{a}z}$$

so that υ_a is a self-inverse automorphism of U taking 0 to a. Define

$$f_a(z) = \left(\frac{\sqrt{1 - |a|^2}}{1 - \bar{a}z}\right)^{2/p}$$

so that $(f_a)^{p/2}$ is the normalized reproducing kernel at a for the Hilbert space H^2. Observe that f_a is in the unit ball of H^p and that as $|a| \to 1^-$, f_a converges uniformly on compact subsets of U to the zero function. Thus $\|Kf_a\|_{H^p} \to 0$ for any compact operator K on H^p.

Using (5.4) and a standard multivalent change-of-variables argument (see [25, section 4.3] or [10, Section 2]), we have

$$\begin{aligned}
\|C_\psi f_a\|_{H^p}^p &= p^2/2\int_U |f_a|^{p-2}|f_a'|^2 N_\psi\, d\nu + |f_a(\psi(0))|^p \\
&= 2|a|^2\int_U \frac{1-|a|^2}{|1-\bar a w|^4} N_\psi(w)\, d\nu(w) + |f_a(\psi(0))|^p \\
&= \frac{2|a|^2}{1-|a|^2}\int_U N_\psi |\upsilon_a'|^2 d\nu + |f_a(\psi(0))|^p \\
&= \frac{2|a|^2}{1-|a|^2}\int_U N_\psi(\upsilon_a)\, d\nu + |f_a(\psi(0))|^p,
\end{aligned}$$

where we remind the reader that N_ψ is the Nevanlinna counting function, defined in (3.1), and $d\nu$ is normalized area measure on U $(dxdy/\pi)$. Now fix $r \in (0,1)$ and choose $|a|$ sufficiently close to 1 so that $\upsilon_a(rU)$ doesn't contain $\psi(0)$. Then $N_\psi \circ \upsilon_a$ has the sub-mean value property on rU (by [10, Section 2]; see also [26, Section 10.6]): $(1/r^2)\int_{rU} N_\psi(\upsilon_a)d\nu \geq N_\psi(\upsilon_a(0)) = N_\psi(a)$. Thus we have

$$\|C_\psi f_a\|_{H^p}^p \geq r^2 \frac{2|a|^2}{1-|a|^2} N_\psi(a).$$

Taking the limit superior as $|a| \to 1^-$ and using the H^2 essential-norm formula [25, Theorem 2.3], we have

$$\limsup_{|a|\to 1^-} \|C_\psi f_a\|_{H^p} \geq [r\|C_\psi|_{H^2}\|_{e_*}]^{2/p}.$$

Now because f_a is in the unit ball of H^p and Kf_a is norm convergent to 0 as $|a| \to 1^-$, we have for any compact operator K on H^p,

$$\begin{aligned}
\|(C_\psi - K)|_{H^p}\|_* &\geq \limsup_{|a|\to 1^-} \|(C_\psi - K) f_a\|_{H^p} \\
&= \limsup_{|a|\to 1^-} \|C_\psi f_a\|_{H^p} \\
&\geq [r\|C_\psi|_{H^2}\|_{e_*}]^{2/p}.
\end{aligned}$$

Since the compact operator K and the number $r \in (0,1)$ are arbitrary, we have (5.5), as desired. □

Theorem 5.4 yields the following generalization of Lemma 5.2.

Corollary 5.5. *Suppose that ψ is an analytic self-map of U with Denjoy-Wolff point $\omega \in \partial U$. Then*

$$r_e(C_\psi|H^p) = r(C_\psi|H^p) = \psi'(\omega)^{-1/p}.$$

Proof. The second equality follows immediately from Lemmas 5.1 and 5.2. To obtain the first, note that

$$\begin{aligned}
r_e(C_\psi|_{H^p}) &= (r_e(C_\psi|_{H^2}))^{2/p} \qquad \text{(Theorem 5.4)} \\
&= (r(C_\psi|_{H^2}))^{2/p} \qquad \text{(Lemma 5.2)} \\
&= r(C_\psi|H^p) \qquad \text{(Lemma 5.1)},
\end{aligned}$$

as desired. □

For a composition operator induced by a self-map with Denjoy-Wolff point inside U, the essential spectral radius certainly need not equal the spectral radius. We have seen that, for example, $r_e(C_\psi|_{H^p}) < 1$ whenever $\psi(0) = 0$ and ψ is a non-inner function that extends analytically across the boundary of U (see Corollary 4.5). On the other hand since 1 is in the point spectrum of every composition operator, the spectral radius of C_ψ is never less than 1.

6. Questions

Recall that φ denotes a holomorphic self-map of the open unit disk U such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, while σ denotes the Koenigs eigenfunction of φ. The principal question raised by our work is whether the sufficient condition (3.8) for σ to belong to H^p is, in general, necessary. We've shown it is necessary when φ is analytic on the closed disk, and as we noted in the Introduction, Pietro Poggi-Corradini has shown it is necessary for univalent φ. In addition, the authors can prove that if φ' extends continuously to $U \cup \{\zeta\}$, where $\zeta \in \partial U$ is a fixed point of φ, then $\sigma \in H^p$ implies

$$|\varphi'(0)|^p \geq \frac{1}{\varphi'(\zeta)}.$$

Hence, our sufficient condition is at least "almost necessary" for a C^1 map φ such that $r_e(C_\varphi|H^2)$ is determined by derivatives of φ at boundary fixed points.

Cowen and MacCluer [8, Corollary 19] recently showed that if φ is univalent then the essential spectrum of $C_\varphi : H^2 \to H^2$ is a (possibly degenerate) disk about the origin, with Koenigs eigenvalues filling out the rest of the spectrum. The same result holds without the hypothesis of univalence if φ is assumed to be analytic on the closed disk (see [7, Theorem 7.36]). Does this "disk + isolated eigenvalues" characterization of the spectrum continue to hold for arbitrary composition operators whose symbols fix a point of U? We remark that for univalently induced composition operators the essential spectral "disk" can indeed degenerate to a single point, even for composition operators that are not power-compact (see [2]).

There are many interesting open problems concerning the geometry of images of Koenigs eigenfunctions. For example, based on the computer-generated image displayed as Figure 2 in Section 1, the Koenigs function for $\varphi(z) = z^5/30 + z^4/30z^4 + 14z/15$ appears to be star-like with respect to the origin. Is this, in fact, the case? More generally, what are necessary and sufficient conditions on φ that will insure $\sigma(U)$ is star-like relative to the origin?

Another question raised by our work is whether

$$\|C_\psi|_{H^p}\|_e = \left(\|C_\psi|_{H^2}\|_e\right)^{2/p},$$

where $\psi : U \to U$ is arbitrary. We've shown in the proof of Theorem 5.4 that

$$\|C_\psi|_{H^p}\|_e \geq \left(\|C_\psi|_{H^2}\|_e\right)^{2/p};$$

moreover, as we pointed out in the preceding section, we can show that for $p \geq 1$,

$$\|C_\psi|_{H^p}\|_e \leq 2^{|1-2/p|} \left(\|C_\psi|_{H^2}\|_e\right)^{2/p}. \tag{6.1}$$

Thus the real issues are (1) whether the inequality (6.1) holds for $0 < p < 1$ and (2) whether the constant $2^{|1-2/p|}$ in the inequality may be replaced by 1.

Remarks. (1) The authors recently learned that P. Poggi-Corradini has shown that the converse of the Main Theorem holds without any extra hypotheses on φ [20].

(2) We would like to thank the referee for providing several references and for offering suggestions that improved the quality of our exposition.

References

1. I. N. Baker and Ch. Pommerenke, *On the iteration of analytic functions in a half-plane II*, J. London Math. Soc. (2) 20 (1979), 255–258.
2. P. S. Bourdon and J. H. Shapiro, *Riesz composition operators*, preprint.
3. S. R. Caradus, W. E. Pfaffenberger, and B. Yood, *Calkin algebras and algebras of operators on Banach spaces*, Marcel Dekker, New York, 1974.
4. J. Caughran and H. J. Schwartz, *Spectra of compact composition operators*, Proc. Amer. Math. Soc. 51 (1975), 127–130.
5. C. C. Cowen, *Iteration and the solution of functional equations for functions analytic in the unit disk*, Trans. Amer. Math. Soc. 265 (1981), 69–95.
6. C. C. Cowen, *Composition operators on H^2*, J. Operator Th. 9 (1983),77–106.
7. C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995.
8. C. C. Cowen and B. D. MacCluer, *Spectra of some composition operators*, J. Funct. Anal. 125 (1994), 223–251.
9. P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1983.
10. M. Essén, D.F. Shea, and C.S. Stanton, *A value-distribution criterion for the class L Log L, and some related questions*, Ann. Inst. Fourier (Grenoble) 35 (1985), 127–150.
11. B. Gramsch, *Integration und holomorphe Funktionen in lokalbeschränkten Räumen*, Math. Annalen 162 (1965), 190–210.
12. L. J. Hanson, *Hardy classes and ranges of functions*, Mich. Math. J. 17 (1970), 235–248.
13. W. K. Hayman, *Multivalent Functions*, Cambridge Tracts in Mathematics #100, second ed., Cambridge University Press, 1994.
14. H. Kamowitz, *The spectra of composition operators on H^p*, J. Funct. Anal. 18 (1975), 132–150.
15. G. Koenigs, *Recherches sur les intégrales de certaines équationes functionelles*, Annales Ecole Normale Superior (3) 1 (1884), Supplément, 3–41.
16. B. D. MacCluer and J. H. Shapiro, *Angular derivatives and compact composition operators on Hardy and Bergman spaces*, Canadian J. Math. 38 (1986), 878–906.
17. E. A. Nordgren, *Composition operators*, Canadian J. Math. 20 (1968), 442–449.
18. P. Poggi-Corradini, *Hardy spaces and twisted sectors for geometric models*, Trans. Amer. Math. Soc., to appear.
19. P. Poggi-Corradini, *The Hardy class of geometric models and the essential spectral radius of composition operators*, preprint.
20. P. Poggi-Corradini, *The Hardy class of Koenigs maps*, preprint.
21. Ch. Pommerenke, *On the iteration of analytic functions in a half-plane I*, J. London Math Soc. (2) 19 (1979), 439–447.
22. W. Rudin, *Real and Complex Analysis*, 3rd edition, McGraw Hill, New York, 1987.
23. J. V. Ryff, *Subordinate H^p functions*, Duke Math. J. 33 (1966), 347–354.
24. H. Schwartz, *Composition operators on H^p*, Thesis: University of Toledo, 1969.
25. J. H. Shapiro, *The essential norm of a composition operator*, Annals of Math. 125 (1987), 375–404.
26. J. H. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York, 1993.
27. J. H. Shapiro, *The Riesz and Fredholm Theories in Linear Topological Spaces*, preprint.
28. J. H. Shapiro, W. Smith, and D. A. Stegenga, *Geometric models and compactness of composition operators*, J. Funct. Anal. 127 (1995), 21–62.
29. G. Valiron, *Sur l'iteration des fonctions holomorphes dans un demi-plan*, Bull des Sci. Math. (2) 55 (1931), 105–128.

ABSTRACT. In 1884, G. Koenigs solved Schroeder's functional equation

$$f \circ \varphi = \lambda f$$

in the following context: φ is a given holomorphic function mapping the open unit disk U into itself and fixing a point $a \in U$, f is holomorphic on U, and λ is a complex scalar. Koenigs showed that if $0 < |\varphi'(a)| < 1$, then Schroeder's equation for φ has a unique holomorphic solution σ satisfying

$$\sigma \circ \varphi = \varphi'(a)\sigma \qquad \text{and} \qquad \sigma'(0) = 1;$$

moreover, he showed that the only other solutions are the obvious ones given by constant multiples of powers of σ. We call σ the Koenigs eigenfunction of φ.

Motivated by fundamental issues in operator theory and function theory, we seek to understand the growth of integral means of Koenigs eigenfunctions. For $0 < p < \infty$, we prove a sufficient condition for the Koenigs eigenfunction of φ to belong to the Hardy space H^p and show that the condition is necessary when φ is analytic on the closed disk. For many mappings φ the condition may be expressed as a relationship between $\varphi'(a)$ and derivatives of φ at points on ∂U that are fixed by some iterate of φ. Our work depends upon a formula we establish for the essential spectral radius of any composition operator on the Hardy space H^p.

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