Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Banach Spaces of Analytic Functions

Proceedings of the Pelczynski Conference
Held at Kent State University,
July 12–16, 1976

Edited by
J. Baker, C. Cleaver, and J. Diestel

Springer-Verlag
Berlin Heidelberg New York 1977

REMARKS ON F-SPACES OF ANALYTIC FUNCTIONS

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1. Introduction. An F-space is a complete, metrizable linear topological space. This article is intended as a case study showing how F-spaces which are not locally convex generate interesting questions which do not occur in the locally convex theory. When these questions are related to concrete spaces of analytic functions they lead to new problems in function theory, and conversely the spaces of analytic functions provide examples that are interesting for the general theory.

The non locally convex phenomenon of interest here is the existence of proper, closed subspaces that are dense in the weak topology. According to the Hahn-Banach theorem such objects never occur in the locally convex theory. In section 2 of this paper we will see that they can occur in F-spaces which are not locally convex: whether they always occur is an open problem. Sections 3 and 4 discuss how these subspaces show up among the shift-invariant subspaces of the Hardy spaces H^p for $0 < p < 1$, and how they can be used to construct interesting examples of F-spaces with trivial dual. The fifth section contains a more detailed examination of some of these matters. In particular there is a fairly complete proof that $H^{\frac{1}{2}}$ contains a proper, closed, weakly dense invariant subspace, and there is a discussion of the "Banach envelope" of $H^p (0 < p < 1)$. The final section deals with the Hardy Algebra N^+, a classical space of analytic functions closely related to the Hardy spaces, whose weakly dense invariant subspaces can be completely characterized.

2. Separation properties. We are going to focus on the way in which continuous linear functionals on an F-space separate points from each other and from closed subspaces. Suppose E is a linear topological space with (topological) dual E'. We say E has the:

a) Point separation property if E' separates points of E, or equivalently if for every $0 \neq x$ in E there is a non-trivial continuous linear functional λ on E with $\lambda(x) \neq 0$;

*I want to thank the Department of Mathematics, University of Wisconsin-Madison for its hospitality during the preparation of this paper.

b) Hahn-Banach separation property if E' separates points of E from closed subspaces not containing them, or equivalently if every closed subspace of E is weakly closed;

c) Hahn-Banach approximation property if each proper, closed subspace of E is annihilated by some non-trivial continuous linear functional, or equivalently if every weakly dense subspace is dense.

It is easy to see that the last two properties can be characterized in terms of quotient spaces; E has the Hahn-Banach separation property iff every quotient of E (by a closed subspace) has the point separation property; and the Hahn-Banach approximation property iff no nontrivial quotient has trivial dual.

Clearly the Hahn-Banach separation property implies the other two. For locally convex (Hausdorff) spaces the Hahn-Banach theorem guarantees all of them, but a space that is not locally convex may have none of them. Perhaps the most famous example of such pathology is the F-space $L^p = L^p([0,1])$ for $0 < p < 1$, taken in its natural metric $d(f,g) = \|f - g\|$, where

$$\|f\| = \int_0^1 |f(t)|^p dt \qquad (f \text{ in } L^p). \tag{2.1}$$

In 1940 M.M. Day [2] showed that L^p has no non-trivial continuous linear functionals, i.e. $(L^p)' = \{0\}$ for $0 < p < 1$. In particular, L^p has none of the above separation properties.

Before we continue, note that the functional $\|\cdot\|$ defined by (2.1) is subadditive on L^p, but not homogeneous. Instead it is p-homogeneous:

$$\|af\| = |a|^p \|f\|$$

for each f in L^p and each scalar a. Such functionals (subadditive, p-homogeneous, vanishing only at the origin) are called p-norms.

An example more subtle than L^p is the sequence space $\ell^p (0 < p < 1)$ with the natural topology induced by the p-norm

$$\|f\| = \sum_1^\infty |f(n)|^p \qquad (f = (f(n))_1^\infty \text{ in } \ell^p).$$

For each positive integer n the evaluation functional

$$\lambda_n(f) = f(n) \qquad (f \text{ in } \ell^p)$$

is continuous, and the family $(\lambda_n)_1^\infty$ separates points of ℓ^p, so ℓ^p has the point separation property. However, ℓ^p does not have the Hahn-Banach approximation property (so it does not have the Hahn-Banach separation property either) when $0 < p < 1$ [27], [30]. This is not difficult to see. First recall the result of Mazur and Orlicz which states that every separable Banach space is isomorphic to a quotient of ℓ^1. Essentially the same proof (let (e_n) be the standard unit vector basis of ℓ^1, choose a countable dense subset (x_n) of the unit ball of the Banach space, extend the bijection $e_n \to x_n$ by linearity and continuity to all of ℓ^1, prove that the extended map is onto) shows that for $0 < p < 1$ every separable p-normed F-space is isomorphic to a quotient of ℓ^p. In particular L^p, which has trivial dual, is isomorphic to a quotient of ℓ^p, so ℓ^p fails to have the Hahn-Banach approximation property by the quotient space characterization.

The closed subspace-call it K- by which ℓ^p was divided to get L^p is an example of a proper, closed, weakly dense subspace in ℓ^p. Unfortunately K is not an easy object to lay hands on: all we get from the proof of its existence is that

$$K = \{f \text{ in } \ell^p : \sum_1^\infty f(n)x_n = 0\}$$

for (x_n) a fixed dense subset of the L^p unit ball (e.g. (x_n) = all trigonometric polynomials with rational coefficients). So other than the fact that K exists, it does not seem to lead to any interesting analysis in ℓ^p.

The situation is different for the Hardy spaces H^p. In 1969 Duren, Romberg, and Shields studied these spaces for $0 < p < 1$ and found proper, closed, weakly dense subspaces <u>invariant under multiplication by the independent variable</u> z. Unlike the subspace K of the previous paragraph, these invariant subspaces of H^p are tractable from the point of view of function theory, and the fact that some of them are weakly dense when $0 < p < 1$ raises new questions about the structure of H^p functions. This matter will be the subject of the next section.

Motivated by these examples, Duren, Romberg, and Shields posed two general questions:

(1) Does every non locally convex F-space fail to have the Hahn-Banach separation property?

(2) Does every non locally convex F-space fail to have the Hahn-Banach approximation property?

The first question was recently answered in the affirmative by N. J. Kalton:

<u>KALTON'S THEOREM</u> [12; Cor. 5.3]. <u>An F-space is locally convex if and only if it has the Hahn-Banach separation property</u>.

An equivalent statement in the language of quotient spaces is this: <u>If an F-space is not locally convex, then some quotient has a dual which does not separate points</u>.

The second question is still open. An equivalent formulation is: <u>does every non locally convex F-space have a nontrivial quotient with trivial dual</u>?

Finally, it should be pointed out that replacing "F-space" by "Hausdorff linear topological space" in these two questions changes the situation drastically. In fact there are many non locally convex linear topological spaces with the Hahn-Banach separation property, and hence also the Hahn-Banach approximation property. For example any real or complex vector space of uncountable Hamel dimension endowed with its strongest vector topology has this property (see [15; p. 53, Problem 6I] and [5; pp. 59-60]). A different class of examples arises from the fact if E is an infinite dimensional Banach space, then E supports a non locally convex topology $\mathcal{T}$ intermediate between its weak and norm topologies [7]. Clearly the non locally convex space (E, $\mathcal{T}$) has the Hahn-Banach separation property (by the Hahn-Banach theorem).

3. Weakly dense invariant subspaces in H^p ($0 < p < 1$). This section introduces the "nice" examples of proper, closed, weakly dense subspaces that occur in the Hardy spaces, and indicates some of the function theoretic problems they suggest. We begin with a brief review of some basic theory: a good reference for this material is the first three chapters of Duren's book [3].

Definition of H^p. The Hardy space H^p is the collection of functions f analytic in the open unit disc Δ such that

$$\|f\|_p^p = \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^p \, dt < \infty.$$

The case usually studied is $1 \leq p < \infty$, where $\|\cdot\|_p$ is a norm which makes H^p into a Banach space. However the interest here is in the range $0 < p < 1$, where the functional $\|\cdot\|_p^p$ is a p-norm which makes H^p into an F-space [3; Page 37, Cor. 2]. That H^p fails to be locally convex was first noticed by Livingston [17]*. For each point z in Δ, the evaluation functional

$$\lambda_z(f) = f(z) \qquad\qquad (f \text{ in } H^p)$$

is continuous [3; Chapter 7, Page 118], and the family (λ_z: z in Δ) separates the points of H^p. Thus H^p has the point separation property, even when $0 < p < 1$.

The boundary correspondence. An important link between H^p theory and real analysis is provided by a theorem of Fatou which states that for each f in H^p the radial limit

$$f^*(e^{it}) = \lim f(re^{it}) \qquad (r \to 1-)$$

exists for almost every t, and the boundary function f^* belongs to $L^p(T)$ (T = unit circle) with

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f^*(e^{it})|^p \, dt = \|f\|_p^p \tag{3.1}$$

[3; Theorem 2.6, P. 21]. In other words the "boundary correspondence" $f \to f^*$ is a

*Regarding this, note that any Banach space can masquerade as a p-normed space: if $\|\cdot\|$ is the norm, then $\|\cdot\|^p$ is a p-norm inducing the same topology.

linear isometry taking H^p onto a closed subspace of $L^p(T)$. The boundary function f^* even retains a vestige of analyticity in that it cannot be too small too often. More precisely, if $0 \not\equiv f \in H^p$, then [3; Theorem 2.2, Page 17]

$$\int_{-\pi}^{\pi} \log|f^*(e^{it})|\,dt > -\infty. \tag{3.2}$$

This implies, for example, that f^* cannot vanish on a subset of T having positive measure.

<u>Inner-outer factorization</u>. Along with some classical function theory, the boundary correspondence yields an important structure theorem for H^p functions. We can ease into this result by noting that for f fixed in H^p, (3.1) and (3.2) imply that $\log|f^*|$ is integrable on T. Thus the formula

$$F(z) = \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log|f^*(e^{it})|\,dt\} \tag{3.3}$$

defines a function F analytic on Δ and without zeroes there. Moreover F belongs to H^p [3; Section 2.4]. Now $\log|F(z)|$ is the Poisson integral of $\log|f^*|$, so standard facts about Poisson integrals and subharmonic functions [3; Chapter 1] show that $\log|f| \leq \log|F|$ on Δ and $\log|f^*| = \log|F^*|$ almost everywhere on T. Thus $q = f/F$ is a function analytic on Δ (since F never vanishes), bounded in modulus by 1, with $|q^*(e^{it})| = 1$ a.e.. Such a function q is called an <u>inner function</u>, and F as given by (3.3) is called an <u>outer function</u>. Thus we see that <u>every H^p function f can be factored as</u>

$$f = qF \tag{3.4}$$

<u>where q is inner and F is outer</u>. It turns out that this factorization is unique up to a multiplicative constant of modulus 1.

A rough interpretation of this inner-outer factorization might go as follows: the outer factor carries the modulus of the boundary function (since $|f^*| = |F^*|$ a.e.), and the inner factor carries the zeroes of the "interior" function (since F never vanishes). But this is not quite the whole story: the inner factor q itself has a further decomposition which is probably best understood through some functional analysis.

<u>Invariant subspaces</u>. Let U denote the operator of "multiplication by z" on H^p, that is:

$$(Uf)(z) = zf(z) \qquad (f \text{ in } H^p,\ z \text{ in } \Delta).$$

U is often called the <u>right shift</u>, or <u>unilateral shift</u> because it shifts the Taylor coefficients of f one unit to the right. U is noteworthy because it is one of the few infinite dimensional bounded operators whose invariant subspace structure has been completely worked out. This was done for $p = 2$ (i.e. Hilbert space) by Beurling, whose theorem forms the basis for a lot of modern interest in H^p spaces.

BEURLING'S THEOREM [1], [11; Ch. 7, p. 99], [10; Lecture II]. A closed subspace of H^p is invariant under U (henceforth just "invariant") if and only if it has the form qH^p for some inner function q.

Clearly each subspace qH^p is invariant, and since $|q^*| = 1$ a.e. it follows from the boundary correspondence that qH^p is closed. So the force of Beurling's theorem lies in the other direction: every invariant subspace is "generated" by an inner function. The cases $p \neq 2$ follow from the case $p = 2$: for $1 \leq p < \infty$ the arguments are given in Helson's monograph [10; Lecture IV], and the case $0 < p < 1$ goes just like $1 \leq p < 2$.

Beurling's theorem can also be viewed as a result on approximation. In this formulation it states that the polynomial multiples of an H^p function form a dense subset of H^p if and only if that function is outer [3; Section 7.3]. In 1969 Duren, Romberg, and Shields added a new dimension to Beurling's result by proving that when $0 < p < 1$ some inner functions (not identically 1) give rise to weakly dense invariant subspaces in H^p. In view of the approximation-theoretic formulation of Beurling's theorem, Duren, Romberg, and Shields called such inner functions weakly outer. It is a challenging unsolved problem in function theory to characterize in a useful way the inner functions that are weakly outer. In particular it is not known if an inner function can be weakly outer for some values of $0 < p < 1$, but not for others.

In order to get some feeling for how the "weakly outer" phenomenon can happen, we need to look at some examples of inner functions and the invariant subspaces they generate.

Examples of inner functions and invariant subspaces.

(a) Blaschke factors. These are functions of the form

$$B(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$$

for α a fixed complex number in Δ. So B is just the conformal mapping of Δ onto itself which interchanges 0 and α. Clearly B is an inner function, and it is easy to see that the invariant subspace BH^p is precisely the collection of H^p functions that vanish at α. More generally, if n is a positive integer, then B^n is an inner function and B^nH^p is just the collection of H^p functions with a zero of order $\geq n$ at α.

(b) Blaschke products. If (z_n) is a sequence of points in Δ which satisfies the Blaschke condition $\Sigma(1 - |z_n|) < \infty$, then the Blaschke product

$$B(z) = \Pi \frac{z_n}{|z_n|} \frac{z_n - z}{1 - \bar{z}_n z}$$

certainly converges at the origin, and in fact it converges uniformly on each compact subset of Δ [3; Theorem 2.4]. Since each factor in the product is an inner function (a Blaschke factor rotated to have positive value at the origin) it seems reasonable to hope that B itself will be inner, and this is exactly what happens. The corresponding invariant subspace BH^p then consists entirely of H^p functions vanishing on the sequence (z_n), with multiplicity at each point $\geq$ the number of times that point occurs in the sequence ($< \infty$ by the Blaschke condition). Moreover, a factorization theorem we will state shortly shows that BH^p consists of <u>all</u> such functions. It follows that <u>for</u> B <u>a Blaschke product</u>, <u>the invariant subspace</u> BH^p <u>is weakly closed in</u> H^p, <u>even when</u> $0 < p < 1$. This is most easily seen when each member of the sequence (z_n) occurs exactly once, for then $BH^p = \bigcap_n \lambda_{z_n}^{-1}(0)$, where the evaluation functionals λ_{z_n} are continuous. For the general case, evaluation of f at z_n is replaced by evaluation of an appropriate derivative of f at z_n (still a continuous linear functional). So no Blaschke product can be weakly outer.

(c) <u>Singular inner functions</u>. There are also inner functions with no zeroes in Δ. Perhaps the most notorious of these is the <u>unit singular function</u>

$$S(z) = \exp\left\{\frac{z+1}{z-1}\right\}$$

which is clearly inner because $\frac{z+1}{z-1}$ is a conformal map taking Δ onto the left half-plane. While the unit singular function has no zeroes in Δ, it has in a certain sense a very strong "boundary zero," in that it decays very quickly to 0 along the unit interval. More precisely:

$$S(r) = O(e^{-1/(1-r)}) \tag{3.5}$$

as $r \to 1-$. Now the growth of each f in H^p is limited by the condition

$$|f(z)| = O((1-r)^{-1/p}) \qquad (r \to 1- ,\ z = re^{it}) \tag{3.6}$$

[3; Theorem 5.9] so each function in the invariant subspace SH^p decays to zero along the unit interval at essentially the same exponential rate as S, hence has the same kind of "boundary zero." We might conjecture from this that SH^p is weakly closed in H^p when $0 < p < 1$; and this is precisely the case (although the proof, which will be indicated in Section 5, proceeds along different lines).

The unit singular function can be generalized in the following way. Let μ be a positive measure on T singular with respect to Lebesgue measure. Then the function

$$S_\mu(z) = \exp\left\{\int \frac{z+e^{it}}{z-e^{it}}\, d\mu(t)\right\}$$

is an inner function (since $\log |S(re^{it})|$ = -Poisson integral of μ, hence is ≤ 0

on Δ, and $\to 0$ a.e. as $r \to 1-$) which has no zeroes in Δ. S_μ is called the singular inner function generated by μ. For example the unit singular function is S_δ, where $d\delta(t)$ = unit mass at $t = 0$.

Singular inner functions are more subtle objects than Blaschke products in that they generate invariant subspaces not associated with zeroes of functions and derivatives. On the other hand every inner function can be factored into a Blaschke product and a singular inner function [3; Theorem 2.8], the factorization being unique up to multiplication by a complex number of modulus one. This result, and the previous factorization (3.4) show that every H^p function can be factored into an inner factor which carries the zeroes (namely the Blaschke product), an outer factor which carries the boundary modulus, and a singular inner factor which has properties in common with both the outer factor (it never vanishes in Δ), and the Blaschke product (it is inner). Of course in order to make these statements correct, we must allow the function $\equiv 1$ to be simultaneously the trivial Blaschke product, singular inner function, and outer function.

The weakly dense invariant subspaces $(0 < p < 1)$. Here at last is the result we are aiming for. The preceding discussion shows that if an inner function is going to be weakly outer (i.e. generate a weakly dense invariant subspace) in $H^p(0<p<1)$, then it cannot have zeroes So by the factorization theorem just stated, it must be a singular inner function. But the unit singular function, which generates a weakly closed invariant subspace, shows that more is needed. Intuitively, the measure generating the unit singular function is so rough that it induces radial decay in the invariant subspace that persists even after weak closure. So the question is: can a measure on T be singular, yet smooth enough that the corresponding singular inner function generates a weakly dense invariant subspace? Duren, Romberg, and Shields [5] proved that such measures exist. In order to state their result properly we need a way of determining the smoothness of a measure.

Definition. The modulus of continuity of a finite positive Borel measure on T is the function

$$\omega_\mu(\delta) = \sup_{|I| \le \delta} \mu(I) \qquad (\delta > 0)$$

where I runs through the intervals of T, and $|I|$ is the length of I.

Note that $\omega_\mu(\delta) = O(\delta)$ implies that μ is absolutely continuous with respect to Lebesgue measure. However it is well known that there exist positive singular measures with any preassigned modulus of continuity rougher than $O(\delta)$; for example $\omega_\mu(\delta) = O(\delta \log \frac{1}{\delta})$ occurs as such a modulus of continuity [24],[4],[9].

The result of Duren, Romberg, and Shields is the following:

THEOREM [5; Theorem 13, Page 53]. Suppose μ is a positive singular measure on T, with modulus of continuity $\omega_\mu(\delta) = 0(\delta \log \frac{1}{\delta})$. Then the invariant subspace $S_\mu H^p$ is weakly dense in H^p, that is, S_μ is weakly outer for $0 < p < 1$.

The proof of this result will be sketched in the fifth section: for the case $p = \frac{1}{2}$ it will be reasonably complete. For the moment it is enough for us to know that H^p contains proper, closed, weakly dense subspaces that are naturally connected with the analysis of the space, when $0 < p < 1$. We have already seen that this leads to an interesting problem in function theory: characterize those singular measures μ whose associated inner functions S_μ are weakly outer. * Now we are going to show that the existence of these "nice" weakly dense subspaces leads to interesting examples in the theory of F-spaces.

4. Some F-spaces with trivial dual. I want to show how the results of the previous section can be used to solve a problem in the general theory of F-spaces. The problem, due to A. Pelczynski, is this: can an F-space with trivial dual have non-trivial compact endomorphisms? Another way of asking this is: can an F-space with no nontrivial finite rank endomorphisms have a nontrivial compact endomorphism? Recently Pallaschke [18] showed that L^p, which has trivial dual, actually has no non-trivial compact endomorphism ($0 < p < 1$), and Kalton [13] generalized this by showing that there is no non-trivial compact operator from L^p into any Hausdorff linear topological space. However, using the existence of proper, closed, weakly dense invariant subspaces in H^p ($0 < p < 1$), Kalton and I were able to show that there exist F-spaces with trivial dual which nevertheless have non-trivial compact endomorphisms [14]. Here is the idea of the proof.

Fix $0 < p < 1$, and let $\varkappa$ denote the topology of uniform convergence on compact subsets of Δ (restricted to H^p). The hero of this section is going to be $\mathcal{T}$, the strongest topology on H^p that agrees on bounded subsets of H^p with $\varkappa$. That is, we declare a set to be $\mathcal{T}$-open if and only if its intersection with every bounded set B is relatively $\varkappa$-open in B. It is easy to check that these $\mathcal{T}$-open sets really are the open sets for a topology, and it is not difficult to see that $\mathcal{T}$ is Hausdorff and weaker than the original topology of H^p. What is not so easy is to check that $\mathcal{T}$ is a vector topology, but fortunately it is. Two other properties of $\mathcal{T}$ which are not difficult to prove are:

(1) The closed unit ball of H^p is $\mathcal{T}$-compact,

and

(2) the invariant subspace qH^p is $\mathcal{T}$-closed for every inner function q. It is the last property that is most critical: we would not be able to prove it if qH^p were not such an explicitly described subspace.

Now choose q to be a singular inner function which generates a weakly dense (proper) invariant subspace qH^p. The existence of such inner functions was discussed in the last section. Then the quotient $E = H^p/qH^p$ is an F-space with

* Done by J. Roberts & B. Korenblum (independently): μ puts no mass on any Carleson set

trivial dual. On the other hand, since qH^p is $\mathcal{T}$-closed, the quotient $F = H^p/qH^p$ in the quotient $\mathcal{T}$-topology is a Hausdorff linear topological space with a topology weaker than that of E (since $\mathcal{T}$ is weaker on H^p than the original topology). In particular, F has trivial dual, and the identity map $E \to F$ is continuous. In fact it is even compact, since the identity map $H^p \to (H^p,\mathcal{T})$ is compact (due to the $\mathcal{T}$-compactness of the H^p unit ball). Thus the map $E + F \to E + F$ defined by $(e,f) \to (0,e)$ is a non-trivial compact endomorphism of $E + F$, and $E + F$ is a linear topological space which has trivial dual, since E and F have trivial dual.

So the fact that the invariant subspace qH^p is weakly dense, but $\mathcal{T}$-closed, leads rather directly to a linear topological space $E + F$ with trivial dual, which nonetheless supports non-trivial compact endomorphisms. Unfortunately $E + F$ is not metrizable (since it turns out that F isn't), but it is not difficult to modify the construction and replace F by a metrizable space (see [14] for the details), yielding an F-space with trivial dual but non-trivial compact endomorphisms. q.e.d.

This gives one instance of how H^p theory can furnish examples that are interesting in the study of F-spaces. Another such example is the quotient space $E = H^p/qH^p$; the first factor in the direct sum mentioned above. We have just seen that E is an F-space with trivial dual, that there is a compact operator taking E into a Hausdorff linear topological space (namely the identity map $E \to F$), and that there is no such operator on L^p. So E is not isomorphic to L^p: in fact E is a new F-space with trivial dual, and should be interesting to study in its own right. For example, does E itself have a compact endomorphism? Does E have a compact convex subset with no extreme point? James Roberts [19] has recently shown that there are linear topological spaces with such subsets: in fact he has shown that L^p is such a space for $0 < p < 1$ [20]. At the moment it is not clear if every linear topological space with trivial dual must contain such a set.

5. The Banach envelope and weakly outer inner functions. This section explores the "weakly outer" phenomenon in greater detail. Surprisingly the controlling interest in the problem belongs to a Banach space of analytic functions that contains H^p.

The Banach envelope. Suppose E is a p-normed F-space ($0 < p \leq 1$) with the point separation property. Then a simple exercise shows that the convex hull of the unit ball of E contains no linear subspaces, so its Minkowski functional [15; Page 15] is a norm on E which induces a topology weaker than the original one (equal to it iff E was locally convex to begin with) but having the same continuous linear functionals. The completion of E in this norm is a Banach space $\hat{E}$ called the Banach envelope of E. Thus E is contained in $\hat{E}$, the inclusion map is continuous, and the dual of E is the same as that of $\hat{E}$ in the sense that the restriction map $\lambda \to \lambda|_E$ takes $\hat{E}'$ onto E'.

It is instructive to follow through this construction for the case $E = \ell^p$ ($0 < p < 1$). Since $\|f\|_1 \leq \|f\|_p$ for all f in ℓ^p we see that ℓ^p is continuously embedded in ℓ^1 via the identity map, so ℓ^1 is a candidate for $\hat{\ell}^p$. In fact $\hat{\ell}^p = \ell^1$ in the sense that the identity map $\ell^p \to \ell^1$ extends to an <u>isometry</u> of $\hat{\ell}^p$ onto ℓ^1. To see this we need only show that the ℓ^1-closure of the convex hull of the ℓ^p-unit ball is the ℓ^1-unit ball. This is easy. Let (e_n) denote the standard unit vector basis for ℓ^1. Then for each f in ℓ^1 the partial sums of the series representation $f = \Sigma f(n)e_n$ are convex combinations of elements in the ℓ^p unit ball, and they converge in ℓ^1 to f.

The fact that E and its Banach envelope have the same dual shows that <u>a subspace of</u> E <u>is weakly dense if and only if it is dense in</u> $\hat{E}$. So when dealing with specific examples, one way to show that a subspace is weakly dense is to get hold of a concrete representation of the Banach envelope and show that the subspace is dense therein. For H^p $(0 < p < 1)$ such a representation was obtained by Duren, Romberg, and Shields:

<u>THEOREM</u> [5; Section 3]. <u>The Banach envelope of</u> H^p <u>is the space</u> B_p <u>of functions</u> f <u>analytic in</u> Δ <u>such that</u>

$$\|f\|_{B_p} = \iint_\Delta |f(z)|(1 - |z|)^{1/p-2}\, dxdy < \infty .$$

Of course the theorem is stated rather loosely: it really means that H^p is contained in B_p, the identity map $H^p \to B_p$ is continuous, and it extends to an isomorphism (not isometry this time) of $\hat{H}^p$ onto B_p. To be precise we should perhaps say that ℓ^1 is an <u>isometric representation</u> of the Banach envelope of ℓ^p, while B_p is merely an <u>isomorphic</u> representation of the Banach envelope of H^p $(0 < p < 1)$. In any case the important thing for our purposes is that <u>a singular inner function</u> q <u>is weakly outer if and only if</u> qH^p <u>(or equivalently the polynomial multiples of</u> q) <u>is dense in</u> B_p. So the weakly outer phenomenon in H^p is also an approximation problem in a certain Banach space of analytic functions. Such problems have been studied extensively by H. S. Shapiro [23], [25], [26], and his results play an important role in our story. Before getting on with this, however, I would like to indicate why B_p is an isomorphic representation of $\hat{H}^p$, at least for $p = \frac{1}{2}$.

<u>$\hat{H}^{\frac{1}{2}} = B_{\frac{1}{2}}$</u>. Note that $B_{\frac{1}{2}}$ is just the subspace of $L^1(\Delta)$ consisting of analytic functions. Our first task is to show that $H^{\frac{1}{2}}$ is contained in $B_{\frac{1}{2}}$ and the identity map is continuous, i.e. that $\|f\|_{B_{\frac{1}{2}}} \le C\|f\|_{\frac{1}{2}}$ for some constant C independent of $f \in H^{\frac{1}{2}}$. This is a result of Hardy and Littlewood [8; Theorem 31]: a short proof can be based on inner-outer factorization and <u>Hardy's inequality</u>.

(5.1) $$\Sigma(n + 1)^{-1}|\hat{f}(n)| \le \pi\|f\|_1$$

for each $f(z) = \Sigma\hat{f}(n)z^n$ in H^1 [3; Page 48].

Since $|\hat{f}(n)| \le \|f\|_1$ for f in H^1, Hardy's inequality yields

$$\frac{1}{\pi}\iint_\Delta |f(z)|^2 \, dxdy = \Sigma(n+1)^{-1}|\hat{f}(n)|^2 \le \pi\|f\|_1^2 \tag{5.2}$$

for f in H^1. Now if $f \in H^{\frac{1}{2}}$ and <u>has no zeroes</u> in Δ, then $f^{\frac{1}{2}} \in H^1$ and $\|f^{\frac{1}{2}}\|_1 = \|f\|_{\frac{1}{2}}^{\frac{1}{2}}$. So replacing f by $f^{\frac{1}{2}}$ in (5.2) we obtain

$$\|f\|_{B_{\frac{1}{2}}} = \iint_\Delta |f(z)| \, dxdy \le \pi^2\|f\|_{\frac{1}{2}}, \tag{5.3}$$

which is the desired inequality. In case f has zeroes in Δ we can still apply this result by using a trick due to Hardy and Littlewood. Let $f = qF$ be the inner-outer factorization (3.4) of f, let $G = (q-1)F$, so $f = G + F$ where neither G nor F have zeroes in Δ, and $\|G\|_{\frac{1}{2}} \le 2\|f\|_{\frac{1}{2}}$; while as we observed in Section 3, $\|F\|_{\frac{1}{2}} = \|f\|_{\frac{1}{2}}$. Applying (5.3) to F and G respectively yields:

$$\begin{aligned}\|f\|_{B_{\frac{1}{2}}} &\le \|G\|_{B_{\frac{1}{2}}} + \|F\|_{B_{\frac{1}{2}}} \\ &\le \pi^2(\|G\|_{\frac{1}{2}} + \|F\|_{\frac{1}{2}}) \qquad \text{(by (5.3))} \\ &\le 3\pi^2\|f\|_{\frac{1}{2}},\end{aligned}$$

which completes the proof that $H^{\frac{1}{2}}$ is contained in $B_{\frac{1}{2}}$ with the identity map continuous.

To complete the proof that $\hat{H}^{\frac{1}{2}}$ is isomorphically represented by $B_{\frac{1}{2}}$ we need to show that the closure in $B_{\frac{1}{2}}$ of the $H^{\frac{1}{2}}$ unit ball contains some ball in $B_{\frac{1}{2}}$. The idea is similar to the one used for ℓ^p, except that instead of using a basis to represent elements of $B_{\frac{1}{2}}$ we use the <u>reproducing kernel</u>

$$K(z,\zeta) = (\beta+1)\,\frac{(1-|\zeta|^2)^\beta}{(1-\bar{\zeta}z)^{\beta+2}}\,.$$

A calculation with power series shows that if $\beta > 0$ and $f \in B_{\frac{1}{2}}$ then

$$f(z) = \iint_\Delta K(z,\zeta)f(\zeta)dA(\zeta) \qquad (z \text{ in } \Delta) \tag{5.4}$$

where $dA(\zeta)$ is normalized Lebesgue area measure on Δ. So if we write $K(\zeta)(z) = K(z,\zeta)$, then each $K(\zeta)$ is an analytic function in z on the closed unit disc, and we can think of (5.4) as representing each f in the $B_{\frac{1}{2}}$ unit ball as a sort of generalized absolutely convex combination of these functions. In fact the right side of (5.4) really is a limit in $B_{\frac{1}{2}}$ of absolutely convex combinations of $K(\zeta)$'s, namely the approximating Riemann sums for the integral (see [28; Section 3] for the details), while a straightforward calculation with $\beta = 2$ shows that $\|K(\zeta)\|_{\frac{1}{2}} \le 3$ for all ζ in Δ. This shows that the $B_{\frac{1}{2}}$ unit ball lies in the $B_{\frac{1}{2}}$-closure of the convex hull of the $H^{\frac{1}{2}}$ ball of radius 3 about the origin, and completes the proof that the Banach envelope of $H^{\frac{1}{2}}$ is (isomorphically represented by) $B_{\frac{1}{2}}$. For other values of p the details of the last half of the proof are similar, with a few more exponents to keep straight [28].

Existence of weakly outer inner functions. We can now outline a proof of the Duren, Romberg, Shields theorem which states that a singular inner function S_μ is weakly outer whenever its modulus of continuity is $O(\delta \log \frac{1}{\delta})$. For simplicity we give the proof only for $p = \frac{1}{2}$, although it is hardly more difficult for the other values of $0 < p < 1$. The argument is due to H. S. Shapiro [26; Theorem 1].

The essential point is that the modulus of continuity condition on the singular measure μ is equivalent to the growth condition

$$(5.5) \qquad m(r) = \min_{|z|=r} |S_\mu(r)| \geq C(1-r)^N$$

for some positive constants C and N independent of $0 \leq r < 1$ (see [23, Theorem 2] for the details). Thus S_μ decays to zero rather slowly near the boundary; and given our remarks about the unit singular function we appear to be on the right track. In fact if $N < 1$ we are finished, since taking p_n to be the n^{th} arithmetic mean of the Taylor series of $1/S_\mu$ we have $p_n \to 1/S$ uniformly on compact subsets of Δ, and [16; Kap. 1, Satz 1, p. 22]

$$\max_{|z|=r} |p_n(z)| \leq \max_{|z|=r} |1/S_\mu(z)| \leq 1/m(|z|) \leq C^{-1}(1-|z|)^{-N}$$

Thus

$$|S_\mu(z)p_n(z)| \leq C^{-1}(1-|z|)^{-N},$$

where the right side is integrable over Δ, and $S_\mu p_n \to 1$ pointwise on Δ. So the Lebesgue Dominated Convergence theorem implies that $S_\mu p_n \to 1$ in $B_{\frac{1}{2}}$. Adopting the notation [E] for the closure in $B_{\frac{1}{2}}$ of the subset E, we see from the above that $[S\, H^{\frac{1}{2}}]$ contains all polynomials. But the polynomials are dense in $B_{\frac{1}{2}}$ [3, Theorem 3] so $[S_\mu H^{\frac{1}{2}}] = B_{\frac{1}{2}}$, and the proof is complete for $N < 1$.

In case $N \geq 1$ note that $S_\mu^{1/2N}$ is the singular inner function induced by the measure $\mu/2N$, and it obeys estimate (5.5) with exponent $= \frac{1}{2}$. So the last paragraph shows that $[S_\mu^{1/2N} H^{\frac{1}{2}}] = B_{\frac{1}{2}}$. Now it is easy to see that if q_1 and q_2 are inner functions with q_2 weakly outer, then $[q_1 q_2 H^{\frac{1}{2}}] = [q_1 H^{\frac{1}{2}}]$, so taking $q_1 = q_2 = S_\mu^{1/2N}$ we obtain

$$\begin{aligned}[S_\mu^{1/N} H^{\frac{1}{2}}] &= [S_\mu^{1/2N} S_\mu^{1/2N} H^{\frac{1}{2}}] \\ &= [S_\mu^{1/2N} H^{\frac{1}{2}}] \\ &= B_{\frac{1}{2}}.\end{aligned}$$

Repeating the argument N times yields $[S_\mu H^{\frac{1}{2}}] = B_{\frac{1}{2}}$. QED

<u>The unit singular function is not weakly outer</u>. We close this section by sketching the proof that the unit singular function $S(z) = \exp \frac{z+1}{z-1}$ generates an invariant subspace that is not weakly dense.

So we have to find a non-trivial continuous linear functional on H^p that vanishes on SH^p. A device for writing down continuous linear functionals on H^p is furnished by the estimate

$$|\hat{f}(n)| \le C\, n^{1/p-1}\|f\|_p \quad , \tag{5.6}$$

where C is independent of the function $f(z) = \Sigma\hat{f}(n)z^n \in H^p$, and $0 < p < 1$ [3; Theorem 6.4, Page 98]. For example if $(a_n)_0^\infty$ is a sequence of complex numbers with

$$|a_n| = O(n^{-1-1/p}) \tag{5.7}$$

then it follows from (5.6) that the formula

$$\lambda(f) = \sum_0^\infty a_n\, \hat{f}(n) \qquad (f \text{ in } H^p) \tag{5.8}$$

defines a continuous linear functional on H^p. We will be done if we can find a non-trivial sequence (a_n) satisfying (5.7) and such that the linear functional λ given by (5.8) annihilates SH^p. The way to do this is to study the function

$$h(e^{it}) = e^{it}(1 - e^{it})^k\, \overline{S(e^{it})} \sim \sum_{-\infty}^{\infty} \hat{h}(n)e^{int} \ ,$$

which can be made to have as many continuous derivatives as desired simply by choosing k sufficiently large ($k = 2n + 1$ guarantees n continuous derivatives, to be precise). Choosing k so that h has at least $1 + 1/p$ continuous derivatives we obtain the estimate $|\hat{h}(n)| = O((|n| + 1)^{-1-1/p})$. Let $a_n = \hat{h}(-n)$ for $n \ge 0$, so (a_n) obeys (5.7), and λ as defined by (5.8) is continuous on H^p. We claim that λ annihilates SH^p. To see this observe that λ can also be expressed by the integral formula

$$\lambda(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(e^{it})h(e^{it})dt$$

at least for f a polynomial. Recalling that for a.e. t we have $|S^*(e^{it})| = 1$, the integral formula for λ gives for each polynomial f:

$$\lambda(Sf) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})(1-e^{it})^k\, e^{it}\, dt = 0 \ .$$

Since the polynomials are dense in H^p [3; Theorem 3.3], the polynomial multiples of S are dense in SH^p, so λ vanishes on all of SH^p. So the proof is complete, provided $a_n = \hat{h}(-n) \ne 0$ for some $n > 0$. If this were not the case then h would be the boundary function of an H^2 function g. Now $e^{it}(1-e^{it})^k$ is the boundary function of the H^2 function $f(z) = z(1-z)^k$, and

$$(Sg)^* = S^*g^* = S^*h = f^* \text{ a.e.},$$

hence the H^2 function SG - f has radial limits zero a.e.. But in Section 3 we remarked (after inequality (3.2)) that a non-trivial H^2 function cannot have 0 as a radial limit on a set of positive measure. Thus $Sg = f$. But this is impossible: for example (3.5) and (3.6) would then guarantee that $f(r) = r(1-r)^k$ tends to zero exponentially fast as $r \to 1-$, which is clearly absurd. Thus $a_n \neq 0$ for some positive n, and the proof is complete.

This argument is due to H. S. Shapiro [25; Theorem 2], who actually proved that S <u>is not weakly outer whenever</u> μ <u>gives positive measure to some thin set</u>. Here a closed subset K of the unit circle is called <u>thin</u> if $\sum \alpha_n \log(1/\alpha_n) < \infty$, where α_n is the length of the n^{th} interval in $T \setminus K$. The case we have just treated is the one where K is a singleton.

6. <u>Weakly dense closed ideals in the Hardy Algebra</u>. In this section we completely characterize the weakly outer inner functions in the Hardy Algebra, a space of analytic functions which is in some sense the "limit" of the H^p spaces as $p \to 0+$.

Let N denote the space of functions f analytic in Δ and having <u>bounded characteristic</u>:

$$\|f\| = \sup_{0 \le r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(1+|f(re^{it})|dt < \infty, \tag{6.1}$$

and let N^+ denote those functions f in N for which the family $\{\log^+|f(re^{it})|: 0 \le r < 1\}$ is uniformly integrable on the unit circle T. N is called the <u>Nevanlinna Class</u>, while N^+ is the <u>Hardy Algebra</u> [6; Ch. 5, Sec. 2] or <u>Smirnov Class</u> [3; Sec. 2.5 and Page 31]. Both N and N^+ are algebras under pointwise multiplication, and $N \supset N^+ \supset H^p$ for all $p > 0$. The metric induced by the subadditive functional $\|\cdot\|$ makes N into a complete space in which N^+ is a closed subspace. Surprisingly this metric does <u>not</u> make N into a linear topological space - the scalar multiplication is discontinuous [29], but fortunately N^+ is spared this embarassment. It turns out that N^+ is a complete topological algebra in the metric induced by $\|\cdot\|$; and that while N^+ is not locally convex, it does have the point separation property (for example it is not difficult to show that the point evaluations $f \to f(z)$ are continuous for each z in Δ).

The Hardy Algebra is often regarded as the limiting case of the H^p spaces because it shares many properties with H^p; in particular the boundary correspondence, the inner-outer factorization, Beurling's theorem, and the density of polynomials (cf., [31; Theorem 4]). Beurling's theorem becomes particularly attractive in this setting. Since N^+ is a topological algebra in which polynomials are dense, a closed subspace is invariant under multiplication by z if and only if it is an <u>ideal</u>. Thus Beurling's theorem for N^+ states that <u>the closed ideals are precisely those of the form</u> qN^+ <u>where</u> q <u>is an inner function</u>, that is, the closed ideals are the principal ideals generated by inner functions.

Since N^+ is not locally convex it is possible that some closed ideals are weakly dense. Using the methods of Section 5 these can be characterized as follows:

THEOREM. A closed ideal in N^+ is weakly dense if and only if it is generated by a singular inner function (i.e. an inner function without zeroes.)

This result can be restated in a couple of ways: an ideal is weakly dense iff it has no common zero in Δ; or an inner function is weakly outer in N^+ iff it is singular. In any case the continuity of point evaluations shows that a closed weakly dense ideal can only be generated by a singular inner function, so the task at hand is to show that every singular inner function does generate such an ideal (recall that the analogous result for H^p was false). This has been done by Roberts and Stoll [21] for the case where the measure associated with the inner function has no point mass.

To prove the theorem we need an analogue for N^+ of the Banach envelope. Since the function $\|\cdot\|$ is not p-homogeneous for any $0 < p \leq 1$ this requires some care, but the idea is still the same. Let C_n denote the convex hull of the N^+ - ball of radius $1/n$ about the origin. It is easy to check that the family (C_n) of convex sets forms a local base for a locally convex metrizable topology on N^+ that is weaker than the original one, but has the same continuous linear functionals [15; Page 109]. The completion $\hat{N}^+$ of N^+ in this topology is a Fréchet space (locally convex F-space) containing N^+ and having the same dual: we might call it the Fréchet envelope of N^+. Just as in the previous section, a subspace of N^+ is weakly dense if and only if it is dense in the Fréchet envelope.

So a concrete representation of $\hat{N}^+$ is needed. This was recently supplied by N. Yanagihara [31] who identified $\hat{N}^+$ with the space F^+ of functions f analytic in Δ such that

$$\sup |f(z)| \exp \{-c/(1 - |z|\} < \infty \qquad (|z| < 1) \tag{6.2}$$

for every $c > 0$. The precise statement is that $F^+ \supset N^+$, and the identity map $N^+ \to F^+$ extends to an isomorphism of F^+ onto $\hat{N}^+$, where F^+ has the topology induced by the seminorms (6.2) as c runs through all positive reals. It is not difficult to see that the seminorms (6.2) can be replaced by an equivalent family

$$\|f\|_c = \iint_\Delta |f(z)| \exp \{-c(1 - |z|)\} \, dxdy \tag{6.3}$$

where c runs through the positive reals (in fact Yanagihara actually defined F^+ in terms of still a third equivalent family of seminorms).

Proof of the Theorem (Cf. [26; Theorem 1]). Let q be a singular inner function. We must show that the ideal qN^+ is dense in F^+, or equivalently that the polynomial multiples of q are dense in the normed space $(F^+, \|\cdot\|_c)$ for each $c > 0$. Since

$|q(z)| = \exp\{-\text{Poisson integral of } \mu\}$, where μ is the singular measure that induces q, and since the Poisson integral of μ at z is $\leq 2\|\mu\|/(1-|z|)$, it follows that the minimum modulus of q is limited by

$$m(r) = \min_{|z|=r} |q(z)| \geq \exp\{-2\|\mu\|/(1-r)\}.$$

Now proceeding as in the last section, let p_n be the n^{th} arithmetric mean of the partial sums of the Taylor series of $1/q$, so $p_n q \to 1$ pointwise on Δ, and for each z in Δ:

$$|q(z)\ p_n(z)| \leq |p_n(z)| \leq 1/m(|z|) \leq \exp\{2\|\mu\|/(1-|z|)\}.$$

so when $c \geq 2\|\mu\|$ the Dominated Convergence Theorem shows that $\|p_n q - 1\|_c \to 0$, hence qN^+ is dense in $(F^+, \|\cdot\|_c)$. For $c < 2\|\mu\|$ the above argument still works on q^ϵ for $0 < \epsilon \leq c/2\|\mu\|$, and an iteration like the one performed in the last section shows that qN^+ is dense in $(F^+, \|\cdot\|_c)$, which completes the proof.

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