Introduction

Our initial setting is a metric space X, which you can, if you wish, take to be a subset of $\mathbb{R}^n$, or even of the complex plane (with the Euclidean metric, of course). These notes discuss ways in which X can be thought of as "being just one piece." We start with what is perhaps the most intuitive such notion—that of *arcwise connectedness.*

1 Arcwise connectedness

1.1 Definition. We say X is *arcwise connected* if any two of its points are joined by an arc that lies entirely in X. More precisely: given $p, q \in X$ there exists a continuous function $\gamma : [a, b] \to X$ such that $\gamma(a) = p$ and $\gamma(b) = q$.

In this definition, $[a, b]$ is a finite, closed subinterval of $\mathbb{R}$, which without loss of generality you may take to be the closed unit interval $[0, 1]$ (Exercise).

1.2 Examples. These are all (mostly simple) exercises:

(a) The image of a continuous curve in X is arcwise connected.

(b) Convex subsets of $\mathbb{R}^n$ are arcwise connected, and more generally so are subsets that are *starlike* with respect to some point.[1]

(c) Suppose X is the union of any pair of nonvoid disjoint open subsets in $\mathbb{R}^n$. Then X is *not* arcwise connected.

(d) The *topologist's sine curve*

$$X = \{(x, \sin\frac{1}{x}) \in \mathbb{R}^2 : 0 < x \leq 1\} \bigcup \{(0, y) \in \mathbb{R}^2 : |y| \leq 1\}$$

is *not* arcwise connected.

2 Connectedness

2.1 Definition. To say that X is *disconnected* means that X is the union of two nonvoid open subsets U and V of X that are disjoint. When this happens, we say the pair $\{U, V\}$ *disconnects* X. If X is not disconnected, we say it is *connected.*

2.2 Examples. Once again, the proofs of these statements are left as exercises.

(a) Every disjoint union of nonvoid open subsets of $\mathbb{R}^n$ is disconnected.

(b) Any finite or countable set in $\mathbb{R}^n$ (e.g., the rationals in $\mathbb{R}$) is disconnected

[1] $X \subset \mathbb{R}^n$ is said to be *starlike* with respect to a point $p \in X$ if for every $q \in X$ the line segment from p to q lies entirely in X.

(c) The Cantor Middle Thirds set—discussed in Math 828, and hopefully, in your undergraduate real analysis course—is disconnected. We will see shortly (in §3.4) that these last two examples are "maximally disconnected."

Here is a useful sufficient condition for connectedness:

2.3 Theorem. *Every arcwise connected metric space is connected.*

PROOF. Suppose X is a metric space that is arcwise connected. We will show that it cannot be disconnected. Suppose to the contrary, there is a pair of nonvoid open sets U and V in X that disconnect it, i.e., whose union is X and whose intersection is empty. Since neither U nor V is empty we can choose a point $p \in U$ and another point $q \in V$. By arcwise connectedness, there is a continuous map $\gamma : [a, b] \to X$ with $\gamma(a) = p$ and $\gamma(b) = q$. Let τ be the supremum of all the numbers $t \in [a, b]$ such that $\gamma(t) \in U$. Then by the continuity of γ and the definition of τ, $\gamma(\tau)$ is a limit point of both U and V. Since $X = U \cup V$, and both U and V are open in X, they are both also closed in X, so $\gamma(\tau)$ belongs to both U and V, contradicting the disjointness of these sets. Thus there can be no such disconnection of X. $\square$

2.4 Corollary. *Every convex, or even starlike, subset of $\mathbb{R}^n$ is connected. In particular, intervals of the real line are connected.*

This corollary, along with the next result, shows that *every continuous curve in a metric space is connected.*

2.5 Theorem. *If X and Y are metric spaces with X connected, and if $f : X \to Y$ is a continuous mapping, then $f(X)$ is connected.*

PROOF. If $f(X)$ is disconnected then there exist open sets $\{U, V\}$ that disconnect it. By the continuity of f both of the sets $f^{-1}(U)$ and $f^{-1}(V)$ are open, hence—as you can easily check—the pair $\{f^{-1}(U), f^{-1}(V)\}$ disconnects X. $\square$

Although arcwise connectedness provides a convenient sufficient condition for connectedness, it is not necessary, as the following result shows.

2.6 Exercise. *The topologist's sine curve of §1.2 is connected.*

That was the bad news: connectedness is not, in general, equivalent to arcwise connectedness. The good news is that for *open* subsets of $\mathbb{R}^n$ the two notions of connectedness *are* equivalent:

2.7 Theorem. *Every open, connected subset of $\mathbb{R}^n$ is arcwise connected.*

PROOF. Suppose $X \subset \mathbb{R}^n$ is open and connected. Fix $x_0 \in X$ (there is nothing to prove if X is empty) and let U be the set of points x in X for which there exists an arc in X from x_0 to x. U is not empty, because it contains x_0 (Why?). We will be done if we can show that $U = X$.

To this end let $V = X \backslash U$. We will show that both U and V are open in X, so because X is connected, V will have to be empty (otherwise the pair $\{U, V\}$ would disconnect X, leading to a contradiction). To see that U is open, note that, because X is open in $\mathbb{R}^n$, each $x \in U$ is the center of an open ball B in $\mathbb{R}^n$ with $B \subset X$. Now every point p of B can be joined to x by a line segment, and by definition, x_0 can be joined to x by an arc γ. Thus x_0 can be joined to p by the arc in X you get by first following γ from x_0 to x and then going from x to p via the line segment.[2] Thus $B \subset U$, so U is open.

A similar argument shows that V is also open; if $V \neq \emptyset$ you fix a point $y \in V$ and let B be an open ball in X centered at y. Claim: $B \subset V$. If not then there would be a point $x \in B$ that lies in $X \backslash V = U$. Then we could join x_0 (the "base point" of U) to x by an arc (definition of U), and then by following the line segment from x to y we could get an arc in X from x_0 to y. This would put y in U, a contradiction. Thus U is open, and the proof is complete. $\square$

2.8 Remark. The proof works word-for-word in any *locally arcwise connected* metric space: I leave it for you to make up a suitable definition for this property, and to observe that the topologist's sine curve fails to have it.

2.9 Connected subsets. Every subset of a metric space is itself a metric space in the original metric. If this new "subset metric space" is connected, we say the original subset is connected. More precisely:

> *A subset S of a metric space X is connected iff there does not exist a pair $\{U, V\}$ of nonvoid disjoint sets, open in the relative topology that S inherits from X, with $U \cup V = S$.*

The next result, a useful sufficient condition for connectedness, is the foundation for all that follows here.

2.10 Theorem. *Let I be any index set and $\{X_i : i \in I\}$ a collection of connected subsets of the metric space X. If $\bigcap_{i \in I} X_i \neq \emptyset$ then $\bigcup_{i \in I} X_i$ is connected.*

PROOF. We may suppose without loss of generality that $X = \bigcup_{i \in I} X_i$. Suppose X is disconnected. Our goal is to show that some X_i is disconnected.

We are given open sets U and V that disconnect X, and a common point p that lies in every X_i. Since $X = U \cup V$ the point p lies in U or V—let's say it's in U. Then $U_i \stackrel{\text{def}}{=} U \cap X_i \neq \emptyset$ for every index i (since $p \in U_i$), and because V isn't empty, $V_i \stackrel{\text{def}}{=} V \cap X_i \neq \emptyset$ for *some* index i. For this lucky index i, the pair $\{U_i, V_i\}$ disconnects X_i. $\square$

[2] Exercise: make this rigorous.

2.11 Corollary. *Suppose E is a connected subset of a metric space X and F a subset of the limit points of E. Then $E \cup F$ is connected; in particular, the closure of E is connected.*

PROOF. By Theorem 2.10 it's enough to prove the result for $F = \{p\}$, a singleton. Suppose p is a limit point of E and $E \cup \{p\}$ is *not* connected. We will show that E is not connected. We are given a pair $\{U, V\}$ of sets that are open in $E \cup \{p\}$, and which disconnect it. So p is in one of those sets—say in U. Then, by disjointness, $V \subset E$.

CLAIM: $U = \{p\}$.

If this were not so then U would intersect E, hence the open sets $U\backslash\{p\}$ and V would disconnect E, contradicting the hypothesis that E is connected.

Thus the singleton $\{p\}$ is (relatively) open in $E \cup \{p\}$, so there is an open subset G of X whose intersection with $E \cup \{p\}$ is just $\{p\}$. This shows that p is not a limit point of E. □

2.12 Exercise. Suppose that the closure of E is connected. Must E itself be connected?

3 Decomposition into Components

In this section we show that even if a metric space is disconnected, it can be written as a disjoint union of maximal connected pieces. These are called *components.*

3.1 Components. If X is a metric space and $x \in X$, let $C(x)$ be the union of all the connected subsets of X that contain x. $C(x)$ is not empty because $x \in C(x)$ (the singleton $\{x\}$ is a connected set), and clearly $C(x)$ is the largest connected subset of X that contains x. Theorem 2.10 tells us that:

(a) $C(x)$ is connected, and

(b) if $y \in X$ does not lie in $C(x)$, then $C(y)$ is disjoint from $C(x)$ (else $C(x) \cup C(y)$ would be a connected set that contains x and is larger than $C(x)$, contradicting the maximality of $C(x)$).

The set $C(x)$ is called the *connected component*, or just the *component*, of x. We have just shown that:

> *Every metric space is the pairwise disjoint union of its components.*

3.2 Exercise. Suppose a pair $\{U, V\}$ of open sets disconnects the metric space X. Show that if $u \in U$ and $v \in V$ then u and v lie in different components of X.

3.3 Examples. Based on the previous exercise, the you can easily supply examples of component decompositions by drawing disjoint collections of arcs and other connected figures in the plane. What is perhaps more interesting is to notice that components can be just single points.

(a) If X is a finite or countable subset of $\mathbb{R}^n$, with the Euclidean metric, then $C(x) = \{x\}$ for any $x \in X$.

(b) The same is true for the Cantor Middle-Thirds set in $[0, 1]$.

Metric spaces in which all the components are single points are called *totally disconnected* spaces. They are, in some sense, the most disconnected spaces possible.[3]

3.4 Theorem. *The components of a metric space are closed.*

PROOF. If C is a component of X then it is the largest connected subset of X that contains any of its points. But Corollary 2.11 asserts that the closure of C is also connected, so by maximality, C equals its closure, and so is itself closed. □

In general the components of a metric space need not be open: the space of rational numbers, in the relative topology of $\mathbb{R}$, furnishes a striking example of this disappointing phenomenon. However, as was the case with arcwise connectedness, open subsets of Euclidean space behave better.

3.5 Theorem. *Suppose G is an open subset of $\mathbb{R}^n$. Then every component of G is open in $\mathbb{R}^n$.*

PROOF. Suppose C is a component of G. Let $p \in C$. Since G is open there exists an open ball B in $\mathbb{R}^n$, centered at p and contained entirely in G. We'll be done if we can show that $B \subset C$. But if this were not the case then $B \cup G$ would be connected by Theorem 2.10, and strictly *larger* than C, thus violating the maximality of C. □

Note that the proof works in any metric space that is *locally connected* in the sense that: for every point of the space, every neighborhood of that point contains a further neighborhood (of the same point) that is connected. For such a space the components are therefore both open and closed.

3.6 Corollary. *An open subset of $\mathbb{R}^n$ can have at most countably many components.*

PROOF. By Theorem 3.5 the components of an open set form a disjoint family of open subsets of $\mathbb{R}^n$. But no such family can be more than countable, since each contains a point, all of whose coordinates are rational, and the collection of such points is countable. □

[3] It's actually possible to be even *more* disconnected than this: A space is called *extremally disconnected* if the closure of every open set is open (or equivalently, if the topology has a basis of sets that are both closed and open). The rationals don't have this property, but any space with no limit points (e.g. a finite set) does have it.

3.7 Remark. *Closed* subsets of $\mathbb{R}^n$ may have uncountably many components, however. The Cantor set in $\mathbb{R}^1$ is such an example; according to §3.3(b)), the only components are the one-point subsets, and there are uncountably many of these.

4 Plane Domains

In this section G always denotes a domain (i.e. and open, connected set) in the complex plane. We study the complement of G in the Riemann Sphere $\hat{\mathbb{C}}$. The component that contains ∞ is, for obvious reasons, called the *unbounded component* of $\hat{\mathbb{C}}\backslash G$. The other components are bounded, and these are called, again for obvious reasons, *holes* (draw some pictures!). Since $\hat{\mathbb{C}}\backslash G$ is closed in $\hat{\mathbb{C}}$, so are its components (by Theorem 3.4 and the fact that a relatively closed subset of a closed set is closed in the ambient space). In particular: *The holes of a plane domain G are closed and bounded, and therefore compact.*

4.1 Exercise. Give examples to show that, if one works only in $\mathbb{C}$, rather than in $\hat{\mathbb{C}}$, a plane domain can have many (even infinitely many) unbounded components. In $\hat{\mathbb{C}}$, of course, all these components would be glued together into one by the point at infinity.

4.2 Boundary. Recall that the boundary of a subset A of a metric space X is the set of points $p \in X$ such that every neighborhood of p contains both points of A and points of $X\backslash A$.

4.3 Lemma. *If C is any component of $\hat{\mathbb{C}}\backslash G$, then*

(a) $\partial C \subset \partial G$, and

(b) $G \cup C$ is connected.

PROOF. (a) Suppose $p \in \partial C$ were not a boundary point of G. Then there would be an open disc Δ centered at p that did not intersect G. But because p is a boundary point of C, Δ intersects some point not in C, thus making $C \cup \Delta$ strictly larger than C, entirely contained in the complement of G, and connected (by Theorem 2.10). But this would violate the maximality of C. Thus every boundary point of C must be a boundary point of G.

(b) By the open-ness of G and part (a), ∂C is a set of limit points of G. Since G is connected, so is $G_1 \stackrel{\text{def}}{=} G \cup \partial C$, by Theorem 2.11. Since C is closed, it contains its boundary, so $G_1 \cup C = G \cup C$, hence the latter set is connected. □

In general, $G \cup C$ needn't be open, as shown by taking G to be the unit disc and removing both the origin and a sequence converging to the origin. Then $C = \{0\}$ is a component of $\hat{\mathbb{C}}\backslash G$, but $G \cup \{0\}$, while connected, is not open. However if we "fill in all the holes" then we *do* get a domain.

4.4 Theorem. *If G is a plane domain then the union of G and all of its holes is again a domain.*

PROOF. By Lemma 4.3 the union of G with any of the bounded components of $\hat{\mathbb{C}}\backslash G$ (a.k.a "holes") is connected. Each of these new sets contains G, and the union of all of them is, by Theorem 2.10, connected. Call this set G_b. Since the complement of G_b in $\hat{\mathbb{C}}$ is the unbounded component of $\hat{\mathbb{C}}\backslash G$, which is closed, we see that G_b is open in $\hat{\mathbb{C}}$, and therefore in G. Thus G_b is a domain. $\square$

At this point we connect up with our textbook:

4.5 Definition. A domain G of the complex plane is called *simply connected* if $\hat{\mathbb{C}}\backslash G$ has just one component (necessarily the unbounded one).

Using this definition we can rephrase Theorem 4.4 as follows:

The union of a domain and all of its holes is a simply connected domain.