CLUSTER SET, ESSENTIAL RANGE, AND DISTANCE ESTIMATES IN BMO

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To my teacher, Allen Shields, for his sixtieth birthday

Introduction. This paper focuses primarily on functions F, holomorphic or harmonic on the open unit disc U, for which the radial limit function

$$F^*(\zeta) = \lim_{r \to 1-} F(r\zeta)$$

exists finitely for almost every $\zeta \in \partial U$. For example, F could be a holomorphic function of bounded characteristic, or it could be the Poisson integral of a measure on the boundary. Our goal is to study the relationship between the *cluster set* of F and the *essential range* of F^*.

Clearly the cluster set contains the essential range. We want to know when they coincide. Obviously they do if F extends continuously to the boundary of U, but consideration of the "unit singular function" $F(z) = \exp\{(z+1)/(z-1)\}$ shows that they need not, even if F is bounded and holomorphic on U.

We are going to prove that cluster set and essential range coincide whenever F is the Poisson integral of a function of *vanishing mean oscillation*. This class contains all harmonic functions which extend continuously to the boundary, some which do not, and even some which are unbounded. Our result shows that every function of vanishing mean oscillation has connected essential range; it recovers the well-known fact that among the inner functions, only the finite Blaschke products can have boundary function of vanishing mean oscillation [21, §3]; and it has consequences for the algebra QC of quasi-continuous functions on the unit circle.

These results emerge from a distance estimate: *If F is the Poisson integral of a function of bounded mean oscillation, then the distance from F^* to the space of functions of vanishing mean oscillation is bounded below by the Hausdorff distance between the cluster set and the essential range* (in plain English: the largest distance by which you can avoid every point of the essential range, while staying in the cluster set). Examples show that this estimate is sharp.

Our proofs work as well for the unit ball B of $\mathbf{C}^n$ with $n > 1$. A feature peculiar to higher dimensions is the fact that the sets on which a holomorphic function has constant value "propagate" to the boundary: every value is a cluster value. Our results therefore show that if F, holomorphic on B, is the Poisson integral of a function of vanishing mean oscillation on ∂B, then the essential range of F^* contains $F(B)$.

Received July 30, 1985. Revision received May 18, 1987.
Research supported in part by the National Science Foundation.
Michigan Math. J. 34 (1987).

Rudin [14, §19.1.12] originally focused attention on this phenomenon by asking if it might be exhibited by the class of *bounded* holomorphic functions on B. If so, then B could not support *inner functions* (nonconstant bounded holomorphic functions F with $|F^*|=1$ a.e. on ∂B). However, thanks to the efforts of Hakim and Sibony [8] and Løw [12], and (independently) of Aleksandrov [1], we now know that inner functions do exist on B. For historical comments and further developments, see [14], [15], and [17].

It is known that the cluster set of an inner function on B must be the entire closed unit disc ([14, §19.1.3]; [15, Thm. 2.1]; [13, §3]), so our results show that the boundary function cannot have vanishing mean oscillation. Such conclusions have been obtained for other classes of functions, not comparable to our "analytic VMO," by Rudin [17, Ch. 18] and Tamm [22].

We also consider the effect of composing a BMO function with (the boundary function of) an inner function. We show that such a composition has the same norm as the original BMO function, but cannot lie in VMO. It must, in fact, be located as far as possible from VMO.

Here is a more detailed outline of the paper. After setting out some terminology and notation in Section 1, we devote Section 2 to a local form of the lower distance estimate (Theorem 2.1), which implies the global one stated above. The third section treats higher-dimensional generalizations, while the final one discusses compositions of BMO functions with inner functions.

I would like to take this opportunity to thank my colleagues Sheldon Axler, Paul Bourdon, Wade Ramey, Paula Russo, and Bill Sledd for many informative discussions about the material in this paper. I am also indebted to the referee, who, in addition to carefully proofreading the paper, called my attention to reference [16] and made a number of thoughtful suggestions which improved the exposition.

1. Preliminaries.

1.0. NOTATION. S denotes the Riemann sphere, U is the unit disc, and σ is normalized Lebesgue measure on ∂U, the unit circle. If A is a subset of S, then the closure of A in S is denoted by $\bar{A}$. The letters I and J, possibly with subscripts, denote closed arcs on ∂U, possibly all of ∂U. The capital letters F, G, and H represent continuous (usually harmonic or holomorphic) functions on U for which the radial limit functions, as defined in the Introduction, exist σ-a.e. on some arc. We denote the radial limit function of F by F^*. Lower-case letters f, g denote measurable functions that are integrable with respect to σ over some arc of ∂U.

1.1. ESSENTIAL RANGE. Suppose f is a (finite) complex-valued measurable function defined a.e. on I. The *essential range* of f over I will be denoted by $R(f, I)$, or just $R(f)$ if $I=\partial U$: it is the set of points $w \in S$ for which

$$\sigma\{f^{-1}(V)\cap I\}>0$$

for every open S-neighborhood V of w.

Finally, if $\zeta \in \partial U$, then $R(f, \zeta)$, the *essential range at* ζ, is defined to be the intersection of all the sets $R(f, J)$ where J runs over all intervals centered at ζ.

Possibly such a J may extend outside I, in which case we take the definition of essential range to be: $R(f, J) = R(f, J \cap I)$. It is easy to check that each $R(f, J)$ is a nonvoid compact subset of S, hence so is $R(f, \zeta)$.

1.2. PROPOSITION. *For I and f as above: $R(f, I) = \bigcup \{R(f, \zeta) : \zeta \in I\}$.*

Proof. It is clear from the definition that $R(f, \zeta) \subset R(f, I)$ for every $\zeta \in I$. Conversely, supose $w \in R(f, I)$. We must find $\zeta \in I$ such that $w \in R(f, \zeta)$. Split the arc I into two closed subarcs J_1 and J_2 of equal length. An easy exercise shows that $R(f, I) = R(f, J_1) \cup R(f, J_2)$, hence w belongs to $R(f, J_1)$ or $R(f, J_2)$. Let I_1 denote the lucky interval and repeat the subdivision process with I_1 in place of I. Keep going, to obtain a nested sequence of closed intervals I_n with lengths tending to zero. These intervals intersect at a point $\zeta \in I$ (I is closed, recall). Let J_n be the smallest interval containing I_n and centered at ζ. The length of J_n is no more than twice that of I_n, so $\bigcap J_n = \{\zeta\}$. Thus $R(f, \zeta) = \bigcap R(f, J_n)$. But $w \in R(f, J_n)$ for every n, so $w \in R(f, \zeta)$. □

1.3. CLUSTER SETS. The *cluster set* $C(F, \zeta)$ of F at ζ is the set of points $w \in S$ such that there exists a sequence (z_n) in U converging to ζ for which $F(z_n) \to w$. Equivalently: if $\Omega(\zeta, r)$ is the intersection of U with the open disc of radius r centered at ζ, then $C(F, \zeta)$ is the intersection of the closures of the sets $F(\Omega(\zeta, r))$ as r ranges through positive values. It follows immediately from this formulation of the definition that $C(F, \zeta)$ is a nonvoid compact subset of the Riemann sphere, which, by the continuity of F on U and by the compactness of the sphere, is connected. The global cluster set $C(F)$ is just the union of the sets $C(F, \zeta)$ as ζ runs over the unit circle. Equivalently, $C(F)$ is the intersection of the closures of the sets $F(A_r)$, where A_r is the annulus $U \backslash rU$ and $0 \le r < 1$. As before, $C(F)$ is a compact, connected subset of S. Clearly $R(F^*, \zeta) \subset C(F, \zeta)$ for each $\zeta \in \partial U$, and $R(F^*) \subset C(F)$. For more material on cluster sets, see [3].

1.4. MEAN OSCILLATION. By L^p we mean $L^p(\sigma)$, where (as above) σ denotes normalized Lebesgue measure on ∂U. For $f \in L^1$ and I a subinterval of ∂U, we write:

$$I[f] = \frac{1}{\sigma(I)} \int_I f \, d\sigma,$$

and if $f \in L^2$, we define:

$$\|f\|_*^2 = \sup I[|f - I[f]|^2],$$

where the supremum ranges over all $I \subset \partial U$. BMO denotes the collection of all f for which $\|f\|_* < \infty$: each such f is said to be of *bounded mean oscillation*. By VMO (for *vanishing mean oscillation*) we denote those functions $f \in \mathrm{BMO}$ for which

$$\lim_{\delta \to 0+} \sup_{\sigma(I) < \delta} I[|f - I[f]|^2] = 0.$$

If we identify functions which differ a.e. by a constant, then $\| \ \|_*$ is a norm making BMO into a Banach space, and VMO is a closed subspace in which the continuous functions on ∂U are dense. Also, BMO contains L^∞, and is contained within L^p for every $0 < p < \infty$ (see [19] and [6] for details).

1.5. GARSIA NORM. For $z \in U$ let μ_z denote the Poisson measure on ∂U for the point z, that is:

$$d\mu_z(\zeta) = \frac{1-|z|^2}{|1-\bar{\zeta}z|^2}\, d\sigma(\zeta).$$

For $f \in L^1$, write $P[f]$ for the Poisson integral of f:

$$P[f](z) = \int_{\partial U} f\, d\mu_z \quad (z \text{ in } U).$$

So $P[f]^* = f$ a.e. on ∂U. For $f \in L^2$, the (possibly infinite) *Garsia norm* $\|f\|_G$ is defined by the equation:

$$\|f\|_G^2 = \sup_{z \in U} P[|f - P[f](z)|^2](z).$$

It is well known that $f \in$ BMO if and only if $\|f\|_G < \infty$; and $f \in$ VMO if and only if $f \in$ BMO and $P[|f-P[f](z)|^2](z) \to 0$ as $|z| \to 1-$. Moreover, the Garsia norm really is a norm on BMO, and it is equivalent to the one defined in the previous section (see [18, Thm. 1] and [19, §4, p. 36]).

The idea behind this equivalence is that if $z = re^{i\theta} \in U$, then $d\mu_z$ is a probability measure on ∂U that behaves very much like the measure $\chi_I\, d\sigma/\sigma(I)$, where I is the subarc of normalized length $1-r$ centered at $e^{i\theta}$, and χ_I is its characteristic function. If this measure is substituted for $d\mu_z$ in the definition of Garsia norm, then the original norm $\| \ \|_*$ results.

The Garsia norm, being conformally invariant, is often better suited to applications involving function theory. For the rest of this paper it is the only norm we will consider on BMO.

1.6. HAUSDORFF DISTANCE. We extend the Euclidean metric to an "infinite-valued metric" on S through the conventions: $|\infty - z| = \infty$ for z finite, and $|\infty - \infty| = 0$.

If B is a closed subset of the Riemann sphere S, and $a \in S$, the (possibly infinite) distance from a to B is:

$$\operatorname{dist}(a, B) = \inf\{|a-b| : b \in B\}.$$

If A is another closed subset of S, write

$$\rho(A, B) = \sup\{d(a, B) : a \in A\}.$$

This quantity answers the question "How far can you get away from every point of B, while remaining in A?" Finally, the *Hausdorff distance* between A and B is defined by:

$$\operatorname{dist}(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

This last definition, when restricted to the compact subsets of the finite plane, results in an honest metric [9, §28, pp. 166–172].

In this paper, A will always be a cluster set, and B the corresponding essential range; hence $B \subset A$, so $\operatorname{dist}(A, B) = \rho(A, B)$. Our results will show that for Poisson integrals of BMO functions, these quantities are actually finite.

2. BMO distance estimate. For $f \in \mathrm{BMO}$, denote by $\mathrm{dist}(f, \mathrm{VMO})$ the distance from f to the closed subspace VMO as measured in the *Garsia norm:*

$$\mathrm{dist}(f, \mathrm{VMO}) = \inf\{\|f-g\|_G : g \in \mathrm{VMO}\}.$$

Here is the main result of this section. We will see in Section 3 that it is sharp.

2.1. THEOREM. *Suppose* $f \in \mathrm{BMO}$ *and* $F = P[f]$. *Then*

$$\begin{aligned}\mathrm{dist}(f, \mathrm{VMO}) &\geq \sup\{\mathrm{dist}(R(F^*, \zeta), C(F, \zeta)) : \zeta \in \partial U\}\\ &\geq \mathrm{dist}(R(F^*), C(F)).\end{aligned}$$

Proof. Recall that $f = F^*$ a.e., that $R(F^*, \zeta) \subset C(F, \zeta)$ for every $\zeta \in \partial U$, and that both are nonvoid, compact subsets of S. Thus if $C(F, \zeta) = \{\infty\}$, then the same is true of $R(F^*, \zeta)$; and the distance between the two sets is zero, by our convention about $|\infty - \infty|$.

So we need only consider those $\zeta \in \partial U$ for which $C(F, \zeta) \neq \{\infty\}$. For such a point ζ, fix a finite point $w \in C(F, \zeta)$ and let $\delta(I) = \mathrm{dist}(w, R(F^*, I))$ (possibly zero), where I is an interval centered at ζ. Thus:

$$\delta(I) \uparrow \mathrm{dist}(w, R(F^*, \zeta)) \quad \text{as length } (I) \downarrow 0, \tag{1}$$

and

$$|f - w| \geq \delta(I) \quad \text{a.e. on } I. \tag{2}$$

Note that since f is finite valued a.e. on ∂U, the quantity $\delta(I)$ is finite. Since $w \in C(F, \zeta)$, there exists a sequence (z_n) in U with $z_n \to \zeta$ and $F(z_n) \to w$. Thus, for each n we obtain from (2):

$$\begin{aligned}P[|f - P[f](z_n)|^2](z_n) &= \int_{\partial U} |f - F(z_n)|^2 \, d\mu_{z_n}\\ &\geq \int_I |f - F(z_n)|^2 \, d\mu_{z_n}\\ &\geq \int_I (|f-w| - |w - F(z_n)|)^2 \, d\mu_{z_n}\\ &\geq (\delta(I) - |w - F(z_n)|)^2 \mu_{z_n}(I)\\ &\to \delta(I)^2 \quad \text{as } n \to \infty,\end{aligned}$$

since $\mu_z(I) \to 1$ as $z \to \zeta$, because ζ is in the interior of I. From this follows the (apparently foolhardy) estimate:

$$\|f\|_G \geq \delta(I),$$

which becomes, upon letting the length of I tend to zero, using (1), and then taking the supremum of the right side of the resulting inequality over all $w \in C(F, \zeta)$:

$$\|f\|_G \geq \mathrm{dist}(R(F^*, \zeta), C(F, \zeta)). \tag{3}$$

Note that (3) asserts that whenever $C(F, \zeta) \neq \{\infty\}$, then the Hausdorff distance from essential range to cluster set is finite, hence $R(F^*, \zeta) \neq \{\infty\}$. In other words, $C(F, \zeta) = \{\infty\}$ if and only if $R(F^*, \zeta) = \{\infty\}$.

In inequality (3) we can replace the norm of f by its distance from VMO by exploiting the translation-invariance of Hausdorff distance, and the density in VMO of the continuous functions on ∂U. To this end, suppose g is continuous on ∂U, and set $G=P[g]$. Then

$$R(F^*-G^*,\zeta)=R(F^*,\zeta)-g(\zeta) \quad \text{and} \quad C(F-G,\zeta)=C(F,\zeta)-g(\zeta).$$

Since the Hausdorff distance is translation invarant,

$$\operatorname{dist}(R(F^*-G^*,\zeta),C(F-G,\zeta))=\operatorname{dist}(R(F^*,\zeta),C(F,\zeta)).$$

Thus f may be replaced by $f-g$ in (3) without changing the right-hand side of the inequality. The first inequality to be proved now follows upon taking the infimum of the left side of the resulting inequality over all functions g continuous on ∂U.

As for the second inequality, suppose $w\in C(F)\setminus R(F^*)$ (if $C(F)=R(F^*)$ then there is nothing to prove). Then $w\in C(F,\zeta)$ for some $\zeta\in\partial U$, and by Proposition 1.2, $w\notin R(F^*,\zeta)$. Thus the various definitions of distance yield:

$$\operatorname{dist}(w,R(F^*))\le\operatorname{dist}(w,R(F^*,\zeta))\le\operatorname{dist}(C(F,\zeta),R(F^*,\zeta)),$$

from which the desired result follows upon taking the supremum of the left-hand side over all $w\in C(F)\setminus R(F^*)$. □

2.2. COROLLARY. *Suppose* $F=P[f]$, *where* $f\in$ VMO. *Then*

(a) *for every* $\zeta\in\partial U$: $C(F,\zeta)=R(F^*,\zeta)$, *hence* $R(F^*,\zeta)$ *is a connected subset of* S.

(b) $C(F)=R(F^*)$, *hence* $R(F^*)$ *is a connected subset of* S.

There is also a result intermediate between (a) and (b) above: If I is a closed subarc of ∂U, then $C(F,I)=R(F^*,I)$. Therefore, since $C(F,I)$ is connected, so is $R(F^*,I)$.

2.3. RANGE VS. ESSENTIAL RANGE. Paul Bourdon has pointed out that the previous results also contain information about a third set associated with the boundary behavior of F, namely its *range* $F^*(\partial U)$, which we define in the obvious way to be the set of all finite or infinite radial limit values $F^*(\zeta)$ where ζ ranges over those points of ∂U at which such limits exist. Clearly:

$$R(F^*)\subset\overline{F^*(\partial U)}\subset C(F). \tag{1}$$

If F is the unit singular function mentioned in the Introduction:

$$F(z)=\exp\{(z+1)/(z-1)\},$$

then $R(F^*)=\partial U$; but $F^*(\partial U)=\overline{F^*(\partial U)}=\partial U\cup\{0\}$, hence the essential range need not coincide with the closure of the range, even for bounded holomorphic functions.

However the inclusions (1) above and Corollary 2.2(b) show that: *if* $f\in$ VMO *and* $F=P[f]$, *then the closure of the range of* F^* *coincides with the essential range of* f, *and both are connected subsets of the Riemann sphere.*

2.4. REMARKS ON QC. The intersection $L^\infty \cap \mathrm{VMO}$ is denoted by QC (for "quasi-continuous"). Sarason [18] observed that QC is a Banach subalgebra of L^∞. Sheldon Axler has pointed out that Corollary 2.2 can be derived for QC functions by Banach algebra methods. The key to his proof is the fact that QC functions, regarded as functions on the maximal ideal space of H^∞ [6, Ch. 5, p. 218, Problem 16], are constant on the support sets of representing measures. Now just as in the case of H^∞, the maximal ideal space of QC can be decomposed into "fibers," one for each point of the unit circle. Axler also points out that Corollary 2.2, along with some standard arguments about restriction algebras, shows that each of these QC-fibers is connected.

2.5. DISCS IN CLUSTER SETS. In [2, Thm. 1.4] Axler and the author showed that if F is a bounded holomorphic function on U then

$$\mathrm{dist}(F^*, \mathrm{VMO}) \leq \sup_{\zeta \in \partial U} \sqrt{\pi^{-1} \operatorname{Area} C(F, \zeta)}\,.$$

This estimate has since been extended by Gamelin [5] to the setting of strictly pseudoconvex domains in $\mathbf{C}^n$, and generalized by Stanton [20] to holomorphic functions that are Poisson integrals of BMO functions. Along with Theorem 2.1 it implies that the cluster set of a bounded holomorphic function F has positive area whenever it differs from the essential range of F^*. In fact much more is true, and in greater generality, as seen in the following.

THEOREM. *Suppose F, holomorphic on U, has a finite radial limit at almost every point of an arc $I \subset \partial U$ with center ζ. Then $\partial C(F, \zeta) \subset R(F^*, \zeta)$.*

This result follows directly from [3, Thm. 5.7, p. 98]. It implies that $C(F, \zeta) \setminus R(F^* \zeta)$ is an open subset of S, hence must have positive area whenever it is non-empty.

2.6. EQUALITY IN THEOREM 2.1. Suppose F is the unit singular function. Clearly F extends continuously to $\partial U \setminus \{1\}$. An elementary mapping exercise shows that it takes: the unit disc into itself, every open arc in ∂U with an endpoint at 1 infinitely often onto ∂U, and $U \cap \{|z-1| < \epsilon\}$ infinitely often onto $U \setminus \{0\}$ for every $\epsilon > 0$. Thus $R(F^*, 1) = \partial U$, and $C(F, 1) = \bar{U}$. So Theorem 2.1 asserts that $\mathrm{dist}(F^*, \mathrm{VMO}) \geq 1$, and the upper distance estimate of [2], stated in Section 2.5, shows that this distance is *exactly* 1. This example shows that equality is possible in Theorem 2.1.

More generally, any inner function F that is not a finite Blaschke product exhibits the same kind of boundary behavior at every point of ∂U over which it cannot by analytically continued ([3, Thm. 5.4, p. 95]; [23, Thm. 7.48, p. 281]). In Section 4 we will give another proof of the fact that every such function lies exactly one Garsia norm unit from VMO, and therefore achieves equality in Theorem 2.1.

3. Higher dimensions. The arguments given in the previous sections for the unit disc carry over almost without change to the setting of the unit ball $B = B_n$

of $\mathbf{C}^n$ for $n>1$. Here we briefly indicate how this goes, and comment on the "propagation" phenomenon that is special to the higher-dimensional situation.

We emphasize that in this section the complex dimension n is strictly larger than 1. Let σ denote normalized surface area measure on ∂B (the unit sphere of $2n-1$ real dimensions). The space BMO = BMO(∂B) is defined exactly as in Section 1.4, except that instead of intervals, I now runs over "non-isotropic caps" in ∂B_n of the form:

$$\{\zeta \in \partial B : |1-\langle \zeta, \zeta_0\rangle| < \delta\} \quad (\zeta_0 \in \partial B \text{ and } 0<\delta<2)$$

(see [14, Ch. 5]). Here $\langle\ ,\ \rangle$ denotes the usual complex Euclidean inner product on $\mathbf{C}^n$. Further information about BMO in this setting can be found in [4], [7], and [11].

The *Poisson–Szegö measure* for $z \in B$ is defined by:

$$d\mu_z(\zeta) = \frac{(1-|z|^2)^n}{|1-\langle z, \zeta\rangle|^{2n}}\, d\sigma(\zeta),$$

and the Poisson–Szegö integral $P[f]$ of a function $f \in L^1$ is defined as in Section 1.5. $P[f]$ is no longer necessarily harmonic in the usual sense, but it is M-harmonic in the sense of Rudin [14, Ch. 4].

With this notation, the Garsia norm of a function f is defined just as in Section 1.5. An argument entirely similar to the one used for the unit disc shows that the Garsia norm characterizes both BMO and VMO, and induces on BMO a norm equivalent to the original one. The relevant estimates for the Poisson kernel (but not the proof of equivalence) can be found in Chapter 5 of [14].

With these conventions, Theorem 2.1 and Corollary 2.2 remain true when U is replaced by B.

To this point we have emphasized the similarities between function theory in the disc and in the ball. Here is a fundamental difference.

3.1. PROPOSITION. *If F is holomorphic on B, then* $\overline{F(B)} = C(F)$.

Proof. If F extends to be a continuous function on the closed unit ball, then $F(B) \subset F(\partial B)$. This result is derived in the Introduction of [17] as an easy consequence of the argument principle for functions of one complex variable, and the simple connectivity of ∂B. For the general case, suppose $0<r<1$ and let F_r be dilate of F defined by $F_r(z) = F(rz)$. The previous result applied to F_r asserts that, for $r>|z_0|$, if $z_0 \in B$ is fixed then the value $F(z_0)$ is assumed by F on the sphere $|z|=r$. Thus $F(z_0) \in C(F)$. So $F(B)$ is contained in $C(F)$, hence the same is true of $\overline{F(B)}$. The definition of cluster set provides the opposite inclusion. □

The abbreviation BMOA (respectively VMOA) is frequently used to denote the collection of functions holomorphic on B which are Poisson–Szegö integrals of functions in BMO (respectively VMO). Proposition 3.1 and the generalization to B of Corollary 2.2 yield the following.

3.2. COROLLARY. *If $F \in$ VMOA, then* $\overline{F(B)} = R(F^*)$.

3.3. COROLLARY. *If F is an inner function on B, then $F \notin$ VMOA.*

Rudin [17, Ch. 18] obtained similar results with BMOA replaced by the "Lumer–Hardy spaces," and VMOA replaced by the closure in these spaces of the holomorphic polynomials. Tamm [22] obtained analogues of Corollaries 3.2 and 3.3 for a class of bounded holomorphic functions which obey an additional mean smoothness condition. Neither of these classes contains, nor is contained in, VMOA. As we mentioned in the Introduction, the existence of inner functions on B shows that Corollary 3.2 does not hold for the class of *bounded* holomorphic functions.

It is well known that if F is an inner function on B, then $C(F,\zeta)=\bar{U}$ for every point $\zeta\in\partial B$ ([14, §19.1.3]; [13, §3]). Our next result shows that a vestige of this property persists under considerably weaker assumptions.

3.4. THEOREM. *Suppose F, holomorphic on B, has a finite nontangential limit $F^*(\zeta)$ at a.e. point $\zeta\in\partial B$. If:*

(a) $|F^*|\geq 1$ *a.e. on* ∂B, *and*

(b) $\overline{F(B)}\cap U\neq\varnothing$,

then $C(F,\zeta)\supset U$ for some $\zeta\in\partial B$.

The proof of this result depends on a one-variable lemma that can be viewed as a "nonlocal" version of [3, Thm. 5.7].

3.5. LEMMA. *Suppose F is holomorphic on U, with:*

(a) $\underline{\lim}_{r\to 1-}|F(r\eta)|\geq 1$ *a.e. on* ∂U, *and*

(b) $\overline{F(U)}\cap U\neq\varnothing$. *Then* $U\subset\overline{F(U)}$.

Proof. Suppose first that F satisfies only hypothesis (a). If $a\in U\backslash\overline{F(U)}$, then the function $G=(F-a)^{-1}$ is bounded on U, so hypothesis (a) yields:

$$|G^*|\leq(1-|a|)^{-1}\quad\text{a.e. on }\partial U.$$

Thus G has the same bound over U. In particular, if $w\in\overline{F(U)}$ then $|w-a|\geq(1-|a|)$; that is, the disc D_a centered at a and tangent to the unit circle is disjoint from $\overline{F(U)}$.

To finish the proof of the lemma, suppose now that F also satisfies hypothesis (b). Then $0\in\overline{F(U)}$. For if not, then by the result above $D_0=U$ must be disjoint from $\overline{F(U)}$, contradicting (b). Once 0 is in $\overline{F(U)}$, then all of $\frac{1}{2}U$ must be there too. For if some point a could belong to $\frac{1}{2}U\backslash\overline{F(U)}$, then D_a could not intersect $\overline{F(U)}$. But by the last step $0\in\overline{F(U)}$, and clearly $0\in D_a$, so D_a *does* intersect $\overline{F(U)}$. The rest of the argument follows by induction, the next step being to conclude that $\frac{3}{4}U\subset\overline{F(U)}$. We omit the details. □

Proof of Theorem 3.4. Fix $w\in\overline{F(B)}\cap U$. Then by Proposition 3.1 there exists a sequence of points (z_k) of B_n such that $F(z_k)=w$, and $|z_k|\to 1$. By selecting a subsequence if necessary, we may assume that (z_k) converges to some point of ∂B_n, which without loss of generality we may assume to be the "east pole" e_1. We are going to show that $C(F,e_1)\supset U$. It will be convenient to adopt the notation: $z=(z',z'')$ for points of $\mathbf{C}^n$, where $z'\in\mathbf{C}^{n-1}$ and $z''\in\mathbf{C}$. For $z'\in B_{n-1}$ write

$$G(\lambda,z')=F(z',\lambda(1-|z'|^2)^{1/2})\quad(\lambda\in U).$$

For z' fixed, $G(\lambda, z')$ is holomorphic for $\lambda \in U$. It should be regarded as the restriction of F to the one-dimensional disc

$$D(z') = \{(z', \lambda(1-|z'|^2)^{1/2}) : \lambda \in U\}.$$

We claim that for a.e. (with respect to volume measure on B_{n-1}) fixed $z' \in B_{n-1}$: the function $G(\lambda, z')$ has radial limit $F^*(z', \omega(1-|z'|^2)^{1/2})$ for a.e. $\omega \in \partial U$.

To this end, let E denote the set of points on ∂B_n at which the nontangential limit of F exists finitely. By hypothesis, $\sigma(E) = 1$. By standard integration techniques [14, §1.4]:

$$\begin{aligned} 1 = \sigma(E) &= \int_{\partial B_n} \chi_E(\zeta', \zeta'')\, d\sigma(\zeta) \\ &= \int_{\partial B_n} \int_0^{2\pi} \chi_E(\zeta', e^{i\theta}\zeta'') \frac{d\theta}{2\pi}\, d\sigma(\zeta) \\ &= \int_{\partial B_n} \int_0^{2\pi} \chi_E(\zeta', e^{i\theta}\sqrt{1-|\zeta'|^2}) \frac{d\theta}{2\pi}\, d\sigma(\zeta) \\ &= \int_{B_{n-1}} \int_0^{2\pi} \chi_E(z', e^{i\theta}\sqrt{1-|z'|^2}) \frac{d\theta}{2\pi}\, d\nu(z'), \end{aligned}$$

where the next-to-last line follows from the translation-invariance of Lebesgue measure on the unit circle and where the last line follows from [14, §14.5, formula (1)], since the integrand depends only on ζ'. Here ν is normalized Lebesgue measure on B^{n-1}, so for a.e. $z' \in B_{n-1}$ (henceforth, the "good" points of B_{n-1}), the inner integral in the last line is 1, hence the section of E at z' defined by

$$E_{z'} = \{\omega \in \partial U : (z', \omega(1-|z'|^2)^{1/2}) \in E\}$$

has full measure in ∂U. Fix one of the good points z', and fix $\omega \in E_{z'}$. Then F has a nontangential limit $F^*(\zeta)$ at $\zeta = (z', \omega(1-|z'|^2)^{1/2})$. Now the linear segment $\{(z', r\omega(1-|z'|^2)^{1/2}) : 0 \le r \le 1\}$ from the center of the disc $D(z')$ to the point ζ on its boundary, lies in a cone in B_n with vertex ζ, so F approaches $F^*(\zeta)$ as z tends to ζ through this segment. Thus $G(\lambda, z')$ has limit $F^*(\zeta)$ as λ approaches ω radially.

So far we have proved that, for every good point z',

$$\lim_{r \to 1-} |G(r\omega, z')| \ge 1 \quad \text{for a.e. } \omega \in \partial U.$$

Now recall the sequence (z_k) of F-preimages of $w \in F(B_n) \cap U$ converging to e_1. For a given k, the projection $z_k' \in B_{n-1}$ lies in the closure of the good points (since, having full measure in B_{n-1}, the good points are dense). Thus, we can alter the sequence (z_k), if necessary, without disturbing its convergence to e_1, so that now z_k' is "good" for each k and $|F(z_k)|$ is still <1. The result is that for each k, the functions $G(\lambda, z_k')$ now satisfy the hypotheses of Lemma 3.5, hence the closures of their images all contain U.

In the language of the function F, we have shown that for each k, the closure of $F(D(z_k'))$ contains U. Since the discs $D(z_k')$ converge to e_1, we obtain the desired result: $C(F, e_1)$ contains U. □

Along with Theorem 2.1 (for the unit ball), this result yields the following.

3.5. COROLLARY. *Suppose* $F \in$ BMOA, *with* $\overline{F(B)} \cap U \neq \varnothing$ *and* $|F^*| \geq 1$ *a.e. on* ∂B. *Then* $\operatorname{dist}(F^*, \text{VMO}) \geq 1$.

4. Composition with inner functions. In this section φ is an inner function on $B = B_n$, where we allow the possibility that n might be 1 (in which case $B = U$). Then φ takes B_n into U, and φ^* acts as a Borel measurable transformation of ∂B_n onto ∂U that respects sets of Lebesgue measure zero. That is, if f and g are measurable functions on ∂B, then $f = g$ a.e. if and only if $f \circ \varphi^* = g \circ \varphi^*$ a.e. In fact, if $\varphi(0) = 0$ then φ^* is measure-preserving, where both domain and range are equipped with the appropriate normalized Lebesgue measure σ [17, Thm. 1.3]. This last fact means that the composition map $f \to f \circ \varphi^*$ is an isometry taking $L^p(\partial U)$ into $L^p(\partial B)$ $(0 < p \leq \infty)$. The next result shows that: (a) this result persists if L^p is replaced by BMO, taken in the Garsia norm; and (b) unless $n = 1$ and φ is a finite Blaschke product, composition with φ^* takes any BMO(∂U) function as far as possible from VMO(∂B).

4.1. THEOREM. *Suppose* φ *is an inner function on* $B = B_n$, *where* $n \geq 1$. *Then for every* $f \in \text{BMO}(\partial U)$:

(a) $f \circ \varphi^* \in \text{BMO}(\partial B)$ *and* $\|f \circ \varphi^*\|_G = \|f\|_G$.

Moreover, if $n > 1$, *or if* $n = 1$ *and* φ *is not a finite Blaschke product, then:*

(b) $\operatorname{dist}(f \circ \varphi^*, \text{VMO}(\partial B)) = \|f\|_G$.

In order to efficiently present the proof of this theorem, we require some preliminaries.

4.2. NOTATION. If $h \in L^1(\partial B)$ and $z \in B$, let $G[h](z)^2 = P[|h - P[h](z)|^2](z)$, where $P[h]$ denotes the Poisson–Szegö integral of h.

Thus each functional $G[\cdot](z)$ is a seminorm on BMO, and the entire collection determines the Garsia norm: $\|h\|_G = \sup\{G[h](z) : z \in B\}$.

The crucial step in the proof of Theorem 4.1 is a lemma well known to inner-function enthusiasts (see, e.g., [16, Thm. 3.1]). We present its proof solely in the interests of completeness.

4.3. LEMMA. *If* $z \in B$ *and* $f \in L^1(\partial U)$, *then:*

(a) $f \circ \varphi^* \in L^1(\partial U)$,

(b) $P[f \circ \varphi^*](z) = P[f](\varphi(z))$, *and*

(c) $G[f \circ \varphi^*](z) = G[f](\varphi(z))$.

Proof. (a) Let $a = \varphi(0)$, and for $\lambda \in U$ set $\psi_a(\lambda) = (\lambda - a)/(1 - \bar{a}\lambda)$. Then ψ_a is a conformal automorphism of U. It is easy to check that composition with ψ_a preserves $L^1(\partial U)$, as well as the class of inner functions. Thus $\varphi_a = \psi_a \circ \varphi$ is an inner function on B which takes the origin of B to the origin of U. By our previous remark, φ_a^* induces a measure-preserving transformation $\partial B \to \partial U$, hence an isometry from $L^1(\partial U)$ to $L^1(\partial B)$. It follows that $\varphi^* = \varphi_a \circ \psi_a$ induces a bounded operator taking $L^1(\partial U)$ into $L^1(\partial B)$. Thus $f \in L^1(\partial U)$ implies $f \circ \varphi^* \in L^1(\partial B)$, as desired.

(b) To prove the identity between Poisson integrals, we first consider f to be a trigonometric polynomial. Since both sides are linear functions of f, it is enough to prove the identity for monomials $f(\lambda)=\lambda^k$, where k is an integer. If $k\geq 0$ then, just as in the proof of [17, Thm. 1.3]:

$$P[f\circ\varphi^*](\lambda)=P[\varphi^{*k}](\lambda)=\varphi^k(\lambda)=P[f](\varphi(\lambda)),$$

where the middle identity expresses the fact that a bounded holomorphic function is the Poisson–Szegö integral of its radial limit function. Upon taking complex conjugates on both sides of this identity, and using the fact that $\overline{\varphi^*}=1/\varphi^*$ a.e. on ∂B, we obtain the same result for $n<0$, hence for all trigonometric polynomials f.

Now if $f\in L^1(\partial U)$, then there is a sequence of trigonometric polynomials f_k which converge to f in the norm of $L^1(\partial U)$. By the first paragraph of this proof, $f_k\circ\varphi^*$ converges to $f\circ\varphi^*$ in $L^1(\partial B)$. Thus the corresponding Poisson–Szegö integrals converge at each point of B:

$$P[f\circ\varphi^*](\lambda)=\lim P[f_k\circ\varphi^*](\lambda)=\lim P[f_k](\varphi(\lambda))=P[f](\varphi(\lambda)),$$

yielding the desired result.

Identity (c) is an immediate consequence of (b). □

Proof of Theorem 4.1. Fix $f\in \mathrm{BMO}(\partial U)$.

(a) Since φ is inner, its image is dense in U (by Proposition 3.1 and Corollary 3.4, for example), so the continuity of $G[f]$ on U yields:

$$\begin{aligned}\|f\circ\varphi^*\|_G&=\sup\{G[f\circ\varphi^*](\lambda):\lambda\in B\}\\&=\sup\{G[f](w):w\in\varphi(B)\}\quad&&\text{[by Lemma 4.3(c)]}\\&=\sup\{G[f](w):w\in U\}&&\text{[density of }\varphi(B)\text{ in }U]\\&=\|f\|_G,\end{aligned}$$

as desired.

(b) By part (a) and the definition of distance to a subspace,

$$\operatorname{dist}(f\circ\varphi^*,\mathrm{VMO}(\partial B))\leq\|f\circ\varphi^*\|_G=\|f\|_G.$$

To prove the opposite inequality, fix $\epsilon>0$, and use the density of $\varphi(B)$ in U to choose $w\in\varphi(B)$ so that $G[f](w)>\|f\|_G-\epsilon$. Suppose for the moment that the complex dimension n of the domain is >1. Then by the propagation phenomenon of Section 3, there exists a sequence (λ_k) of points in B with $|\lambda_k|\to 1$, and $\varphi(\lambda_k)=w$ for each k.

Fix a function g continuous on ∂B. By the seminorm nature of G, there is a reverse triangle inequality:

$$G[f\circ\varphi^*-g](\lambda_k)\geq G[f\circ\varphi^*](\lambda_k)-G[g](\lambda_k)$$

for each k. Since g is continuous on ∂B, its Poisson–Szegö integral extends it continuously to $\bar{B}$ [14, Thm. 3.3.4], hence $G[g](\lambda_k)\to 0$ as $k\to\infty$. This fact and the last inequality show that:

$$\begin{aligned}\|f\circ\varphi^*-g\|_G &\geq \limsup G[f\circ\varphi^*-g](\lambda_k)\\ &= \limsup G[f\circ\varphi^*](\lambda_k)\\ &= G[f](\varphi(\lambda_k)) \qquad\qquad \text{[by Lemma 4.3(c)]}\\ &= G[f](w)\\ &\geq \|f\|_G-\epsilon.\end{aligned}$$

Since ϵ is an arbitrary positive quantity, we conclude that $\|f\circ\varphi^*-g\|_G \geq \|f|_G$ for every continuous function g on ∂B. Upon taking the infimum over all such functions g, and recalling that they form a dense subset of VMO, we obtain the desired result when the dimension n is larger than 1.

In case $n=1$ and φ is not a finite Blaschke product, then the same proof works without change, since the values taken on infinitely often by φ form a dense subset of U [3, Thm. 2.1.4, p. 35]. □

We observed in Section 2.5 that if φ is an inner function on U that is not a finite Blaschke product, then φ^* lies exactly one Garsia norm unit away from VMO. Theorem 4.1 yields the following direct proof.

4.4. COROLLARY. *Suppose φ is an inner function on B. Then:*
(a) $\|\varphi^*\|_G=1$, *and*
(b) *if $n>1$, or if $n=1$ and φ is not a finite Blaschke product, then* $\operatorname{dist}(\varphi^*, \mathrm{VMO}(\partial B))=1$.

Proof. Let u denote the identity function on ∂U: $u(\zeta)=\zeta$. In view of Theorem 4.1, we need only show that $\|u\|_G=1$. We leave it to the reader to show that, for $w\in U$, $G[u](w)=1-|w|^2$, whereupon:

$$\|u\|_G=\sup\{G[u](w): |w|<1\}=1. \qquad □$$

REFERENCES

1. A. B. Aleksandrov, *Existence of inner functions in the unit ball* (Russian), Mat. Sb. (N.S.) 118(160) (1982), 147–163; Math. USSR-Sb. 46 (1983), 143–159.
2. S. Axler and J. H. Shapiro, *Putnam's theorem, Alexander's spectral area estimate, and VMO,* Math. Ann. 271 (1985), 161–183.
3. E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets,* Cambridge Univ. Press, Cambridge, 1966.
4. R. R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables,* Ann. of Math. (2) 103 (1976), 611–635.
5. T. W. Gamelin, *On an estimate of Axler and Shapiro,* Math. Ann. 272 (1985), 189–196.
6. J. Garnett, *Bounded analytic functions,* Academic Press, New York, 1981.
7. J. Garnett and R. H. Latter, *The atomic decomposition for Hardy spaces in several complex variables,* Duke Math. J. 45 (1978), 815–845.
8. M. Hakim and N. Sibony, *Fonctions holomorphes bornées sur la boule de* $\mathbf{C}^n$, Invent. Math. 67 (1982), 213–222.
9. F. Hausdorff, *Set theory,* Chelsea, New York, 1957.
10. E. Hille, *Analytic function theory,* Vol. II, Ginn, Boston, Mass., 1962.

11. S. G. Krantz, *Holomorphic functions of bounded mean oscillation and mapping properties of the Szegö projection,* Duke Math. J. 47 (1980), 743–761.
12. E. Løw, *A construction of inner functions on the unit ball in* $\mathbf{C}^p$, Invent. Math. 67 (1982), 223–229.
13. R. M. Range, *On the modulus of boundary values of holomorphic functions,* Proc. Amer. Math. Soc. 65 (1977), 282–286.
14. W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Springer, New York, 1980.
15. ———, *Inner functions in the unit ball of* $\mathbf{C}^n$, J. Funct. Anal. 50 (1983), 100–126.
16. ———, *Composition with inner functions,* Complex Variables Theory Appl. 4 (1984), 7–19.
17. ———, *New constructions of functions holomorphic in the unit ball of* $\mathbf{C}^n$. Conf. Board Math. Sci., Amer. Math. Soc., Providence, R.I., 1986.
18. D. Sarason, *Functions of vanishing mean oscillation,* Trans. Amer. Math. Soc. 207 (1975), 391–405.
19. ———, *Function theory on the unit disc,* Notes for lectures given at a Conference at Virginia Polytechnic Inst. and State Univ., Blacksburg, Va., 1978.
20. C. S. Stanton, *Counting functions and majorization for Jensen measures,* Pacific Math. J. 125 (1986), 459–468.
21. D. Stegenga, *Bounded Toeplitz operators on* H^1 *and applications of the duality between* H^1 *and the functions of bounded mean oscillation,* Amer. J. Math. 98 (1976), 573–589.
22. M. Tamm, *Sur l'image par une fonction holomorphe bornée du bord d'un domaine pseudoconvexe,* C.R. Acad. Sci. Paris Sér. I. Math. 294 (1982), 537–540.
23. A. Zygmund, *Trigonometric series,* Vol. II, 2nd ed., Cambridge Univ. Press, New York, 1959.

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