Notes on the complex exponential and sine functions (§1.5)

I. Periodicity of the imaginary exponential. Recall the definition: if $\theta \in \mathbb{R}$ then $e^{i\theta} \stackrel{\text{def}}{=} \cos\theta + i\sin\theta$. Clearly $e^{i(\theta+2\pi)} = e^{i\theta}$ (because of the 2π periodicity of the sine and cosine functions of ordinary calculus). It's also clear—from drawing a picture of $e^{i\theta}$ on the unit circle, say—that 2π is the "minimal period" of the imaginary exponential, in the sense that if $a \in \mathbb{R}$ has the property that $e^{i(\theta+a)} = e^{i\theta}$ for every $\theta \in \mathbb{R}$ then $a = 2\pi n$ for some integer n.

II. Periodicity of complex the exponential. Recall the definition: if $z = x + iy$ where $x, y \in \mathbb{R}$, then

$$e^z \stackrel{\text{def}}{=} e^x e^{iy} = e^x(\cos y + i\sin y).$$

It's clear from this definition and the periodicity of the imaginary exponential (§I) that $e^{z+2\pi i} = e^z$, i.e.: "The complex exponential function is periodic with period $2\pi i$."

The first thing we want to show in these notes is that the period $2\pi i$ is "minimal" in the same sense that 2π is the minimal period for the imaginary exponential (and for the ordinary sine and cosine).

THE "MINIMAL PERIOD THEOREM" FOR THE COMPLEX EXPONENTIAL. *If $\alpha \in \mathbb{C}$ has the property:*

(1) $$e^{z+\alpha} = e^z \quad \textit{for all } z \in \mathbb{C},$$

then $\alpha = 2\pi ni$ for some integer n.

PROOF. If we set $z = 0$ in (1) we see that $e^\alpha = e^0 = 1$. Now write α in cartesian form: $\alpha = a + ib$ where $a, b \in \mathbb{R}$. Then $1 = e^\alpha = e^a e^{ib}$. Take absolute values on both sides of this last equation to obtain $1 = e^a$, so, (because in this last equation we are dealing with the ordinary exponential of calculus) $a = 0$. Thus $\alpha = ib$, hence our previous equation $e^a = 1$ becomes: $e^{ib} = 1$. It follows from the work of §I that $b = 2\pi n$ for some integer n. Thus $\alpha = 2\pi in$, as promised. □

COROLLARY. $e^\alpha = 1 \iff \alpha = 2\pi ni$ *for some integer n.*

PROOF. If $e^\alpha = 1$ then for each $z \in \mathbb{C}$:

$$e^{z+\alpha} = e^z e^\alpha = e^z$$

so α is a period of the complex exponential, and hence, by the Theorem, is an integer multiple of $2\pi i$. □

III. Univalence[1] of the complex exponential. *The complex exponential is univalent in any open horizontal strip of width 2π (or less).*

Remark. Width 2π is the best we can hope for by the periodicity of the complex exponential noted in §II above.

PROOF OF THEOREM. We'll do a little better, and show that: *if*

$$e^z = e^w \text{ with } |\operatorname{Im} z - \operatorname{Im} w| < 2\pi, \tag{2}$$

then $z = w$.

Thus, for example, if $a \in \mathbb{R}$ and S is the (non-open) strip $\{z \in \mathbb{C} : a < \operatorname{Im} z \le a + 2\pi\}$, then the complex exponential is univalent on S. Also, if S is any open ribbon-shaped region of vertical width 2π or less (draw a picture!), then the complex exponential is univalent on S.

So suppose z and w are complex numbers that satisfy condition (2). We wish to show $z = w$. Multiply both sides of $e^z = e^w$ by e^{-w} and use the addition law for the complex exponential to get $e^{z-w} = 1$, whereupon the Corollary to the "Minimal Period Theorem" of §II insures that $z - w = 2\pi ni$ for some integer n. Thus $\operatorname{Im} z - \operatorname{Im} w = 2\pi n$, hence $|\operatorname{Im} z - \operatorname{Im} w| = 2\pi|n|$. But our hypothesis is that $|\operatorname{Im} z - \operatorname{Im} w| < 2\pi$, hence $n = 0$, whereupon $z = w$. □.

III. Zeros of the complex sine function. Recall that the complex sine function is defined, for $z \in \mathbb{C}$, as:

$$\sin z \stackrel{\text{def}}{=} \frac{e^{iz} - e^{-iz}}{2i} \ .$$

The goal of this section is to show that this extension of the usual sine function of calculus to the complex plane does not add any new zeros.

THEOREM. $\sin z = 0 \iff z = n\pi$ *for some integer* n.

PROOF. By trigonometry we know that $\sin \pi n = 0$ for any integer n, so what's at stake here is the converse: if $\sin z = 0$ then $z = \pi n$ for some integer n.

Well, $\sin z = 0$ implies that $e^{iz} = e^{-iz}$, so by multiplying both sides by e^{iz} and using the addition formula for the complex exponential, we see that $e^{i2z} = 1$, whereupon, by §I, there's an integer n such that $2z = 2\pi n$, i.e., $z = n\pi$. □

IV. Periodicity of the complex sine function. *The minimal period of the complex sine function is 2π.*

PROOF. We know that the complex sine function has period 2π (because of the $2\pi i$ periodicity of the complex exponential). The important assertion here is that if, for some complex number α,

$$\sin(z + \alpha) = \sin z \quad \text{for all } z \in \mathbb{C}, \tag{3}$$

[1]Here "univalence" means "one-to-one-ness". Also, I'll use "univalent" to mean "one-to-one".

then α is an integer multiple of 2π.

So suppose we have (3) for some $\alpha \in \mathbb{C}$. Then upon setting $z = 0$ we see that $\sin\alpha = 0$, hence by §III we know that α is an integer multiple of π. We wish to show that this integer is *even*. In any case, we now know α is *real*. Now set $z = \pi/2$ in (3). Then $\sin(\alpha + \frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$, so (since α is real, hence we're now operating in the realm of ordinary trigonometry):

$$\alpha + \frac{\pi}{2} = \frac{\pi}{2} + 2\pi n \quad \text{for some integer } n$$

whereupon $\alpha = 2\pi n$, as promised. □

V. Univalence of the complex sine function (cf. page 50). *The complex sine is univalent on vertical the strip*

$$V \stackrel{\text{def}}{=} \{z \in \mathbb{C} : 0 < \operatorname{Re} z < \frac{\pi}{2}\}.$$

PROOF. Suppose $z, w \in V$ and $\sin z = \sin w$.

To show: $z = w$.

Well, from the definition of the complex sine we know that

$$e^{iz} - e^{i(-z)} = e^{iw} - e^{i(-w)},$$

so upon rearranging to get "minus powers" on the same side of the equation:

$$e^{iz} - e^{iw} = e^{i(-z)} - e^{i(-w)} = e^{i(-z)}e^{i(-w)}\left(e^{iw} - e^{iz}\right). \tag{4}$$

So either

$$e^{iz} - e^{iw} = 0 \tag{5}$$

or we can divide both sides of (4) by $e^{iz} - e^{iw}$ to yield:

$$-1 = e^{i(-z)}e^{i(-w)} = e^{-i(z+w)}. \tag{6}$$

Suppose it's (5) that's true. Then $e^{iz} = e^{iw}$ so by a now-familiar argument (using the addition formula for the complex exponential, and the result of §II), $z = w + 2\pi n$ for some integer n, i.e. $|\operatorname{Re} z - \operatorname{Re} w| = 2\pi|n|$. But $|\operatorname{Re} z - \operatorname{Re} w| < \pi$, so $n = 0$, hence $z = w$.

I claim (6) can't happen! If it did, then we'd have $e^{i\pi} = -1 = e^{-i(z+w)}$, so by a now-familiar argument, we'd get

$$1 = e^{-i(z+w+\pi)},$$

whereupon the "Corollary" in §I would guarantee that $z + w + \pi = 2\pi n$ for some integer n, i.e. that $z + w$ would be an odd (possibly negative) multiple of π. Thus the same would be true of $\operatorname{Re} z + \operatorname{Re} w$: it would be an odd multiple of π. But since $z, w \in V$ we know $0 < \operatorname{Re} z + \operatorname{Re} w < \pi$, so no such "odd multiple" can exist.

Thus it's only (5) that can happen, and so our theorem is proved. □