Cauchy, Liouville, and the Fundamental Theorem of Algebra

These notes supplement the lectures of Wed. 11/09/ and Fri. 11/11/05.

1. Cauchy's formula for derivatives.

Recall from previous adventures:

1.1. **Cauchy's formula.** *Suppose γ is a simple, closed, piecewise smooth, positively oriented curve in the plane, and Ω is its inside domain. Suppose f is analytic on an open set containing γ and Ω. Then for every $z \in \Omega$:*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \ . \tag{1}$$

Let's differentiate both sides of this equation with respect to z.

$$f'(z) = \frac{1}{2\pi i} \frac{d}{dz} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_\gamma \frac{d}{dz}\left[\frac{f(\zeta)}{\zeta - z}\right] d\zeta = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \ , \tag{2}$$

where the interchange of derivative and integral is justified by the smoothness of f. [1]

Now we can use the same argument on the result of equation (2) to obtain:

$$f''(z) = \frac{2}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^3}\, d\zeta \ ,$$

and more generally, for n any positive integer,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \ .$$

1.2. **Example.** Suppose γ surrounds the point 2 but not the origin. Then, letting $f(z) = 1/z$ in our Cauchy formula for the deriviative (and writing z for ζ, and 2 for z):

$$\int_\gamma \frac{1}{z(z-2)^2}\, dz = 2\pi i f'(2) = 2\pi i(-1/4) = -\pi i/2.$$

[1] You study such interchanges of limiting operations in more rigorous analysis courses like Math 320, Math 421, Math 428-9.

1.3. **Liouville's Theorem.** *Bounded entire functions must be constant.*

Proof. Suppose f is an entire function that is bounded. Thus there's a positive number M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$.

We'll use our Cauchy formula for f' to show that $f' \equiv 0$ on $\mathbb{C}$, which will give the desired result: $f \equiv$ constant on $\mathbb{C}$.

To this end, fix $z \in \mathbb{C}$, and for $R > 0$ consider the positively oriented circle γ_R of radius R, centered at z:

$$\gamma_R(\theta) = z + Re^{i\theta} \qquad (0 \leq \theta \leq 2\pi).$$

By the Cauchy formula for derivatives:

$$\begin{aligned} f'(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + Re^{i\theta})}{(Re^{i\theta})^2}\, iRe^{i\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(z + Re^{i\theta})}{Re^{i\theta}}\, d\theta \end{aligned}$$

Now put absolute values around the result of this calculation and crash them through the integral sign to obtain:

$$|f'(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(z + Re^{i\theta})|}{R}\, d\theta \;\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{M}{R}\, d\theta = \frac{M}{R}\ ,$$

where for the third inequality we have used the fact that $|f| \leq M$ everywhere on $\mathbb{C}$.

Summarizing: for fixed $z \in \mathbb{C}$,

$$|f'(z)| \leq \frac{M}{R} \quad \text{for every } R > 0.$$

Take the limit on both sides of the last inequality as $R \to \infty$. You see that $|f'(z)| = 0$, i.e., $f'(z) = 0$, as desired. □

1.4. **The Fundamental Theorem of Algebra.** *Every non-constant polynomial with real or complex coefficients has a zero in* $\mathbb{C}$.

Proof. Let p be a non-constant say of degree $n > 0$. Thus

$$p(z) = a_0 + a_1 z + \ \cdots\ a_n z^n \quad \text{with} a_n \neq 0.$$

We want to show that $p(z) = 0$ for some $z \in \mathbb{C}$.

Suppose otherwise. Then since p is an entire function with no zero in the plane, its reciprocal $f = 1/p$ is also entire.

Claim: *f is bounded.*

If we can prove this, then by Liouville's Theorem, we'll know $f \equiv$ constant on $\mathbb{C}$, hence $p = 1/f \equiv$ constant on $\mathbb{C}$, contradicting our assumption that p is *not* constant. So p must have had a zero somewhere in $\mathbb{C}$.

Proof of Claim*:* Factor out z^n from p to obtain

$$f(z) = \left(\frac{1}{z^n}\right)\left(\frac{1}{\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + a_n}\right) \qquad (z \in \mathbb{C}).$$

As $z \to \infty$, the denominator of the second term in round brackets converges to $a_n \neq 0$, hence the second term itself goes to $1/a_n$. But the first term tends to zero, hence

$$\lim_{z\to\infty} f(z) = 0.$$

In particular, $|f|$ is bounded by 1 outside of some circle $|z| = R$. Inside this circle $|f|$ is continuous, hence bounded (by something). Thus $|f|$, and therefore f itself, is bounded on the whole complex plane. This proves the Claim, and therefore the Theorem. □

1.5. **Corollary.** *If p is a polynomial of degree $n > 0$ then there exist n complex numbers $z_1, z_2, \ldots z_n$ (not necessarily all distinct) and a complex number $c \neq 0$ such that*

$$p(z) = c(z - z_1)(z - z_2) \cdots (z - z_n) \qquad (z \in \mathbb{C}).$$

Proof. By the Fundamental Theorem, p has a zero, call it z_0, somewhere in the plane. By our previous work on the zeros of analytic functions, this means that p can be factored as

$$p(z) = (z - z_0)p_1(z) \qquad (z \in \mathbb{C}),$$

where p_1 has to be a polynomial of degree $n - 1$. If $n = 1$ then p_1 is constant and we're done. Otherwise repeat the argument on p_1: it factors as $p_1(z) = (z - z_1)p_2(z)$ where p_2 is a polynomial of degree $n-2$. If $n = 2$ we're done. Keep going to obtain the desired factorization of p.[2] □

[2] We're doing, in a very informal way, an induction argument here.