Review Questions for Exam 3

1. State the definition of

   (a) Open set

   (b) Limit point

   (c) Closed set

   Give examples of each.

2. Show that every finite intersection of open sets is open, but that this fails if "finite" is omitted.

   State and prove the corresponding results for closed sets.

3. True or False: If false give counterexample. If true, give a brief proof.

   (a) If a set of real numbers is not open, it's closed.

   (b) If a set is closed then each of its points is a limit point.

   (c) If a set is open then each of its points is a limit point.

   (d) If a series converges then its set of partial sums is closed.

   (e) If a series *diverges* then its set of partial sums is *not* closed.

4. State the definition of "compact set" and use this definition to prove that every compact set is closed and bounded.

5. Which of the following sets are compact (give reasons)?

   (a) The closed unit interval $[0, 1]$.

   (b) The set of irrational numbers that belong to $[0, 1]$

   (c) The union of the set of partial sums of the series $\sum_{n=0}^{\infty} 2^{-n}$ and the one-point set $\{2\}$.

6. True or false (usual rules apply):

   (a) Every bounded countable set is compact.

   (b) Every finite set is compact.

   (c) The set of partial sums of a convergent series is compact.

7. State the definition of "$\lim_{x \to c} f(x) = L$."

   Use this definition to show that if $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ then $\lim_{x \to c} f(x)g(x) = LM$.

8. Use the definition of limit to show that

   (a) $\lim_{x \to 0} \frac{x+1}{x+2} = \frac{1}{2}$

   (b) $\lim_{x \to c} \sqrt{x} = \sqrt{c}$

9. Use the definition of limit to show that if f and g are defined on $\mathbb{R}$ with $\lim_{x \to c} f(x) = 0$, and g is bounded in a neighborhood of c, then $\lim_{x \to c} f(x)g(x) = 0$.

10. Find $\lim_{x \to 0} \sqrt{x}\, T(x)$, where T is Thomae's function.

11. Show that $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

12. Define: "f is *uniformly continuous* on A."

13. Show that $\sin \frac{1}{x}$ is not uniformly continuous on $(0, 1]$.

    Is $\sin \frac{1}{x}$ uniformly continuous on $[10^{-23}, 1]$?

14. Show that uniformly continuous functions take Cauchy sequences to Cauchy sequences.

15. Suppose f is uniformly continuous on $(0, 1]$. Must f be bounded on $(0, 1]$? Prove or give a counterexample.

16. Suppose f is continuous on $(a, b]$, and that $\lim_{x \to a+} f(x)$ exists.

    Show that f is *uniformly continuous* on $(a, b]$.

    Does the converse hold (i.e. does UC imply existence of limit)? If so, prove it. If not, give a counterexample.

17. True or false (usual ground rules):

    (a) f continuous on $[0, 1]$ implies $f([0, 1])$ closed.

    (b) f continuous on $[0, 1]$ implies $f([0, 1])$ bounded.

    (c) f continuous on $(0, 1)$ implies $f((0, 1))$ open.

    (d) f continuous on $(0, 1]$ implies $f((0, 1])$ bounded.

    (e) f continuous on an interval I implies $f(I)$ an interval.

    (f) $f(I)$ an interval implies f continuous on I.