
Review Questions for Exam 1

1. State the Definition of "upper bound", "least upper bound." Use this format:
" $b \in \mathbb{R}$ is an *upper bound* for $A \subset \mathbb{R}$ " means: "For every $a \in A \dots$."
2. What does it mean for two sets to have the same cardinality? Give the formal definition!
3. Define: " $\lim_{n \rightarrow \infty} a_n = a$."
4. State the Axiom of Completeness (AoC).
Give an example of a bounded set of rational numbers that has no *rational* least upper bound. State your definition entirely in terms of rational numbers.
Give an example of a bounded set of real numbers whose least upper bound does not belong to the set.
Show how the AoC implies the Archimedean property of the reals (for every real number there is a larger natural number).
Show how the Archimedean property of the reals implies the density of the rationals in the reals.
5. Show that the set of odd natural numbers is countable.
Show that the set of odd integers (both positive and negative) is countable.
Show that the union of two countable sets is countable.
6. Show that any finite real interval (a, b) (with $a < b$) has the same cardinality as $(0, 1)$.
Show that $(0, 1)$ has the same cardinality as \mathbb{R} .
Show that $(0, \infty)$ has the same cardinality as $(0, 1)$.
Show that any finite (open) real interval has the same cardinality as \mathbb{R} .
7. Show that if A and B are sets of real numbers, both bounded above, and if c is any real number, then $(\sup A) - c = \sup(A - c)$, where $A - c$ is the set of differences $a - c$ where a runs through A .
8. State the Nested Interval Theorem (NIT), and show how the AoC is used to prove it.
Give examples to show that neither of the hypotheses "closed" nor "finite" can be omitted from the NIT.
Show that the NIT fails for \mathbb{Q} .
Show how the NIT is used to prove the uncountability of \mathbb{R} .

9. Show, in “two essentially different ways,” that $\mathbb{R} \neq \mathbb{Q}$.
10. Use intuition to decide that $\lim_{n \rightarrow \infty} \frac{2n^2 - 1}{5n^2 + 2n}$ should exist, and to guess what the limit should be.
 Then use the *definition* of limit to prove that the limit exists, and is what you think it should be.
 Finally, use the *algebraic theorems* about limits, and the fact that $1/n \rightarrow 0$, to prove that the limit exists, and to find it.
11. Use the *definition* of limit to show:
 Limits, when they *exist*, are *unique* (i.e. if $a_n \rightarrow a$ and $a_n \rightarrow a'$, then $a = a'$).
 Every convergent sequence of real numbers is *bounded*.
 Every sequence of non-zero real numbers that converges to a nonzero limit is *bounded away from zero* (i.e., if $a_n \rightarrow a \neq 0$ and $a_n \neq 0$ for all n , then there exists $c > 0$ such that $|a_n| \geq c$ for all n).
12. Show that if $a_n \rightarrow a$ and $b_n \rightarrow b$ then $a_n b_n \rightarrow ab$.
 Show that if $b_n \rightarrow b \neq 0$ then $\frac{1}{b_n} \rightarrow \frac{1}{b}$.
 Show that if $a_n \rightarrow a$ and $b_n \rightarrow b \neq 0$ then $\frac{a_n}{b_n} \rightarrow \frac{a}{b}$.
13. Give examples of sequences (a_n) and (b_n) of real numbers for which:
- $(a_n + b_n)$ converges, but (a_n) does not,
 - $(a_n b_n)$ converges, but (a_n) does not.
14. Suppose the sequences $(a_n + b_n)$ and (a_n) both converge. What can you say about (b_n) and why?
 Formulate and prove a similar result for (a_n/b_n) .
15. Let $x_1 = 1$ and $x_{n+1} = 1 + \frac{1}{2}x_n$ for $n = 1, 2, \dots$. Show by induction that the sequence (x_n) so defined is strictly increasing, that each x_n is < 2 , and that each x_n is rational.