Notes on Uniform Continuity

These notes supplement the discussion in our text on uniform continuity. It shows you one of the important applications of uniform continuity which, informally stated, asserts:

> *Every function that's uniformly continuous on a dense subset has a continuous extension to the whole set.*

To make this statement precise, let's recall that, for a set A of real numbers, a subset $B \subset A$ is said to be *dense in A* if, for every point $a \in A$, every ε-neighborhood $V_\varepsilon(a)$ of a contains a point of B.

For example, the set of rational numbers are dense in the set of reals.

Density can be expressed in terms of sequences as follows—I leave the verification to you:

Sequential characterization of density *A subset B of A is dense in A if and only if for every $a \in A$ there exists a sequence $(b_n) \subset B$ with $b_n \to a$.*

Our main result is the following generalization of the results of Exercise 4.4.13 on p.120. After its proof we'll see an application to one of the most intriguing functions in all of Analysis, the so-called "Devil's Staircase."

The Continuous Extension Theorem. *Suppose f is uniformly continuous on a dense subset B of A. Then there is a unique function F continuous on A such that $F(b) = f(b)$ for every $b \in B$.*

Terminology. Whenever a function $F : A \to \mathbb{R}$ coincides on a subset B of A with a function $f : B \to \mathbb{R}$ we say "F is an extension of f to A." Thus the Continuous Extension Theorem can be restated like this:

> *If f is uniformly continuous on a dense subset B of A then f has a unique continuous extension to A.*

Proof of Uniqueness. Suppose F and G are two continuous extensions of f from B to A. Fix $a \in A$; we want to show that $F(a) = G(a)$. If $a \in B$ this is clear from the definition of "extension." So suppose $a \notin B$. Then, by the "sequential characterization of density" there is a sequence $(b_n) \subset B$ with $b_n \to a$. By the "sequential characterization of continuity,"

$$F(a) = \lim_n F(b_n) = \lim_n f(b_n) = \lim_n G(b_n) = G(a),$$

so the functions F and G are identical on A. ///

Proof of Existence. We'll show that for each $a \in A$ the limit of $f(b)$ exists as $b \to a$ through B. This limit will be our value $F(a)$.

To this end, fix $a \in A$ and let $(b_n) \subset B$ converge to A (density of B in A again).

Claim. *$(f(b_n))$ is a Cauchy sequence.*

Let's accept this Claim for a moment and see where it leads. First, it leads to the fact that $\lim_n f(b_n)$ exists; call it L. But to show that $\lim_{b \to a, b \in B} f(b)$ exists we need to show that if (b'_n) is any other sequence in B that converges to a then $\lim_n f(b'_n) = L$ as well.

This is easy: we know from what we just did that $\lim_n f(b'_n)$ exists; call it L'. We want to show that $L = L'$.

Now $b_n - b'_n \to a - a = 0$ by the limit algebra theorem for sequences. Let $\varepsilon > 0$ be given. Use the uniform continuity of f on B, to choose $\delta > 0$ so that

$$|b - b'| < \delta \quad \text{and} \quad b, b' \in B \quad \Longrightarrow \quad |f(b) - f(b')| \leq \varepsilon. \tag{1}$$

Now use the fact that $b_n - b'_n \to 0$ to choose $N \in \mathbb{N}$ so that

$$n \geq N \quad \Longrightarrow \quad |b_n - b'_n| < \delta \quad \Longrightarrow \quad |f(b_n) - f(b'_n)| < \varepsilon,$$

where the last implication follows from (1). Thus $f(b_n) - f(b'_n) \to 0$. But also

$$f(b_n) - f(b'_n) \to L - L'$$

by "limit algebra," so (uniqueness of limits) $L' - L = 0$, as desired.

It remains to prove the Claim. Let $\varepsilon > 0$ be given. Again use the uniform continuity of f to choose $\delta > 0$ so that (1) holds. Being convergent, (b_n) is a Cauchy sequence, so we can find $N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad |b_n - b_m| < \delta \quad \Longrightarrow \quad |f(b_n) - f(b_m)| < \varepsilon$$

(where does the last implication come from?). Thus the sequence $(f(b_n))$ is Cauchy, hence convergent, and the Claim is proved.

So far: We've shown so far that if f is uniformly continuous on the dense subset B of A then we can produce an extension F on A by:

$$F(a) \stackrel{\text{def}}{=} \lim_{b \to a, b \in B} f(b) \qquad (a \in A), \tag{2}$$

Remains to show: *F is continuous on A.*

Proof of continuity. In fact we'll show F is *uniformly continuous* on A. Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ so that

$$a, a' \in A \quad \text{and} \quad |a - a'| < \delta \quad \Longrightarrow \quad |f(a) - f(a')| < \varepsilon.$$

Once again, choose $\delta > 0$ so that (1) holds. I claim this δ also works for F. To see why, fix a and a' in A with $|a - a'| < \delta$. Choose sequences (b_n) and (b'_n) from B with $b_n \to b$ and $b'_n \to b'$. Then we can choose $N \in \mathbb{N}$ so that

$$(3) \qquad n \geq \mathbb{N} \quad \Longrightarrow \quad |b_n - b'_n| < \delta$$

(make sure you can explain why this is possible). Now the definition of F, limit algebra, the continuity of the absolute value function, (3), and the "order limit theorem" all combine to guarantee that

$$|F(a) - F(a')| = |\lim_n f(b_n) - \lim_n f(b'_n)| = |\lim_n [f(b_n) - f(b'_n)]| = \lim_n |f(b_n) - f(b'_n)| \leq \varepsilon$$

which completes the proof that F is uniformly continuous on A.* ///

The Cantor Function & the "Devil's Staircase." Recall from §3.1 of our textbook the definition of the *Cantor set:* $C \stackrel{\text{def}}{=} \cap_n C_n$, where: C_0 is the closed unit interval $[0, 1]$, C_1 is C_0 with the open middle third removed, C_2 is C_1 with the open middle third of each of its intervals removed, ... and having constructed C_n, consisting of 2^n closed intervals, each of length $1/3^n$, we get C_{n+1} by removing the open middle third from the intervals that comprise C_n.

We observed in class (see also §3.1 of our textbook) that C is compact, uncountable, and of "length zero." Now we want to observe that

$[0, 1]\backslash C$ *is dense in* $[0, 1]$.

To see why this is so, fix $n \in \mathbb{N}$. Since the set C_n is a disjoint collection of closed intervals each of length $1/3^n$, each of its points lies within $1/3^n$ of a point of its complement, hence within $1/3^n$ of a point of $[0, 1]\backslash C$ (since $[0, 1]\backslash C_n \subset [0, 1]\backslash C$).

Since $C \subset C_n$ it follows that each point of C lies, for each $n \in \mathbb{N}$, within $1/3^n$ of a point of $[0, 1]\backslash C$. This also (trivially) true for each point of $[0, 1]\backslash C$, hence each point of $[0, 1]$ is the limit of a sequence of points drawn from $[0, 1]\backslash C$. Thus $[0, 1]\backslash C$ is dense in $[0, 1]$ by the sequential characterization of density. ///

We construct the Cantor function f by first defining it on $[0, 1]\backslash C$. Let $f \equiv 1/2$ on $(1/3, 2/3)$, the interval we removed to make C_1 from C_0. now consider the two intervals $(1/9, 2/9)$ and $(7/9, 8/9)$ we removed from C_1 to make C_2. Define f have constant value $1/4$ on the first of these intervals, and $3/4$ on the second. Keep on going: having defined f on $[0, 1]\backslash C_n$, consider the 2^n intervals of length $1/3^{n+1}$ each that we removed to define C_{N+1}. Define f to take constant value $1/2^{n+1}$ on the first of these, $3/2^{n+1}$ on the next one, ... , etc. So you get a function f defined on $[0, 1]\backslash C$.

Before you look at the next page, draw the graph of the first few stages of the construction of f and convince yourself that for each $n \in \mathbb{N}$:

$$(4) \qquad x, y \in [0, 1]\backslash C \quad \& \quad |x - y| < 1/3^n \quad \Longrightarrow \quad |f(x) - f(y)| < 1/2^n$$

*Why am I allowed to get away with "$\leq \varepsilon$" here, when the definition of limit seems to demand the stronger inequality "$< \varepsilon$?"

Thus f *is uniformly continuous on* $[0,1]\backslash C$, since given $\varepsilon > 0$ we need only choose $n \in \mathbb{N}$ so that $1/2^n \leq \varepsilon$, and then choose $\delta = 1/3^n$. By (4) we then have

$$x, y \in [0,1]\backslash C \quad \& \quad |x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon,$$

as desired.

Now the "Continuous Extension Theorem" guarantees that f has a unique continuous extension to $[0,1]$. This extension is what we call the "Cantor function" in honor of Georg Cantor, who first described its construction. Its graph is sometimes called the "Devil's Staircase."

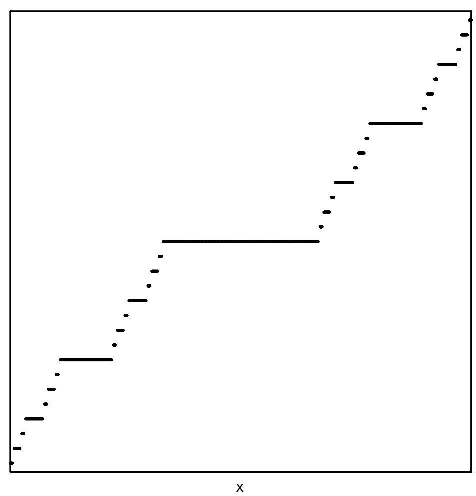

Figure 1: The Devil's Staircase on $[0,1]\backslash C_n$, $n =$?

Exercises. Throughout: f is the Cantor function just constructed on $[0,1]$.

1. What is $f(0)$? $f(1)$? Justify your answers.

2. Show that f is monotone increasing on $[0,1]$, i.e., that if $0 \leq x \leq y \leq 1$ then $f(x) \leq f(y)$.

3. Show that the *modulus of continuity* $\omega(\delta)$ of f is $\leq \delta^\alpha$, where $\alpha = \log 2/\log 3$.

   Suggestion: Use (4) above to get this for $[0,1]\backslash C$; then take limits.

4. Recall from Calculus that each function continuous on a closed bounded interval is Riemann integrable on that interval. So $\int_0^1 f(x)\,dx$ exists. What is it?

   Suggestion: Use the symmetry of the graph $y = f(x)$.

5. Figure 1 shows the graph of the Cantor function defined on $[0,1]\backslash C_n$. What is n for this picture?