Exercises on Uniform Continuity

For $f : A \to \mathbb{R}$ define, for each $\delta > 0$:

$$\omega_A(\delta) = \sup\{|f(x) - f(y)| : x, y \in A, |x - y| < \delta\}$$

The function so defined, $\omega_A : (0, \infty) \to [0, \infty)]$, is called the *modulus of continuity of f over A.*

Exercise 1. Show that f is uniformly continuous on A if and only if $\lim_{\delta \to 0} \omega_A(\delta) = 0$.

Exercise 2. Prove, for f and A as given below (all discussed in class), and $\delta > 0$, the indicated estimates on $\omega_A(\delta)$. Discuss the implications for uniform continuity.

(a) $f(x) = |x|$ and $A = \mathbb{R}$: Show $\omega_A(\delta) \leq \delta$.

(b) $f(x) = x^2$ and $A = [0, 1]$ Show $\omega_A(\delta) \leq 2\delta$.

(c) $f(x) = \sqrt{x}$ and $A = [0, \infty)$. Show $\omega_A(\delta) \leq \sqrt{\delta}$.

(d) $f(x) = x^2$ and $A = [0, \infty)$. Show $\omega_A(\delta) = +\infty$ (we define the supremum of a set that's not bounded above to be $+\infty$).

Exercise 3. Find the modulus of continuity of the function $\sin(1/x)$ over the interval $(0, 1)$, and use your result to discuss the uniform continuity of this function over that set.

Exercise 4. Show that, in the modulus-of-continuity estimates of Exercise 2, the inequalities are all actually *equality.*