Decimal Expansions of Real Numbers

This handout presents an important application that does not seem to appear in our textbook.

The Decimal Representation Theorem. *If x is a real number with $0 \le x < 1$ then*

$$x = \sum_{n=1}^{\infty} \frac{d_n}{10^n} \ , \tag{1}$$

where for each d_n is an integer between 0 and 9 for each $n \in \mathbb{N}$.

Remarks. (a) The Theorem asserts that x has the decimal expansion $.d_1 d_2 d_2 d_3 \ldots$.

(b) By the Geometric Series Theorem (Example 2.7.5) and the Comparison Test (Theorem 2.7.4) there is no doubt that the series on the right-hand side of (1) converges to a number in $[0, 1]$ (you should write out the details of this argument carefully). Thus our goal is to show that *every number in* $[0, 1)$ has such an expansion.

Proof. Fix a number $x \in [0, 1)$. Divide the interval $[0, 1)$ into ten equal subintervals:

$$I_0 = [0, \frac{1}{10}),\ I_1 = [\frac{1}{10}, \frac{2}{10}),\ \ldots\ ,\ I_9 = [\frac{9}{10}, 1).$$

Since the intervals are pairwise disjoint (different ones have empty intersection) and their union is $[0, 1)$, the number x lies in exactly one of these: I_{d_1}. Thus $x_1 = x - \frac{d_1}{10}$ is non-negative and $< \frac{1}{10}$.

Now repeat this bisection-selection process with x_1 in place of x and the interval $[0, \frac{1}{10})$ in place of $[0, 1)$.[1] You see that there's a unique integer d_2 between 0 and 9 for which x_1 lies in the interval $[\frac{d_2}{100}, \frac{d_2+1}{100})$. Thus

$$x_2 \ = \ x_1 - \frac{d_2}{100} \ = \ x - \frac{d_1}{10} - \frac{d_2}{100}$$

lies between 0 and $\frac{1}{100}$.

An induction, which I leave to you, shows that there's a sequence $(d_1, d_2, \ldots)$ of integers between 0 and 9 such that for each $N \in \mathbb{N}$ we have

$$0 \ \le \ x - \sum_{n=1}^{N} \frac{d_n}{10^n} \ < \ 10^{-N} \ ,$$

[1] Or, perhaps more elegantly, apply the bisection-selection procedure to $10x_1$ and the original interval $[0, 1)$.

and this shows the sequence of partial sums of the series (1) converges to x. Thus the series converges to x. ☐

Exercise 1. State and prove a similar theorem for *binary expansions* $.b_1b_2b_3\ldots$, where each b_n is either 0 or 1.

Exercise 2. State and prove a similar theorem for *ternary expansions* $.t_1t_2t_3\ldots$, where each t_n is either 0, 1, or 2.

Exercise 3. Use the Decimal Representation Theorem proved above to state and prove a Decimal Representation Theorem that applies to any real number.

Exercise 4. How does our Decimal Representation Theorem represent the number $.1459999\ldots$? How does it represent any number in $[0, 1)$ whose decimal representation ends in all 9's?