1 Introduction

Our setting is a compact metric space X which you can, if you wish, take to be a compact subset of $\mathbb{R}^n$, or even of the complex plane (with the Euclidean metric, of course). Let $C(X)$ denote the space of all continuous functions on X with values in $\mathbb{C}$ (equally well, you can take the values to lie in $\mathbb{R}$). In $C(X)$ we always regard the distance between functions f and g in $C(X)$ to be

$$\text{dist}\,(f,g) = \max\{|f(x) - g(x)| : x \in X\}.$$

It is easy to check that "dist" is a metric (henceforth: the "max-metric") on $C(X)$, in which a sequence is convergent iff it converges uniformly on X. Similarly, a sequence in $C(X)$ is Cauchy iff it is Cauchy uniformly on X. Thus the max-metric, which from now on we always assume to be part of the definition of $C(X)$, makes that space complete. These notes prove the fundamental theorem about compactness in $C(X)$:

1.1 The Arzela-Ascoli Theorem *If a sequence $\{f_n\}_1^\infty$ in $C(X)$ is* bounded *and* equicontinuous *then it has a uniformly convergent subsequence.*

In this statement,

(a) "$\mathcal{F} \subset C(X)$ is bounded" means that there exists a positive constant $M < \infty$ such that $|f(x)| \leq M$ for each $x \in X$ and each $f \in \mathcal{F}$, and

(b) "$\mathcal{F} \subset C(X)$ is equicontinuous" means that: for every $\varepsilon > 0$ there exists $\delta > 0$ (which depends *only* on ε) such that for $x, y \in X$:

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall f \in \mathcal{F},$$

where d is the metric on X.

1.2 Exercise. The Arzela-Ascoli Theorem is the key to the following result: *A subset $\mathcal{F}$ of $C(X)$ is compact if and only if it is closed, bounded, and equicontinuous.*

1.3 Exercise. You can think of $\mathbb{R}^n$ as (real-valued) $C(X)$ where X is a set containing n points, and the metric on X is the *discrete metric* (the distance between any two different points is 1). The metric thus induced on $\mathbb{R}^n$ is equivalent to, but (unless $n = 1$) not the same as, the Euclidean one, and a subset of $\mathbb{R}^n$ is bounded in the usual Euclidean way if and only if it is bounded in this $C(X)$. Show that every bounded subset of this $C(X)$ is equicontinuous, thus establishing the Bolzano-Weierstrass theorem as a generalization of the Arzela-Ascoli Theorem.

2 Proof of the Arzela-Ascoli Theorem.

STEP I. We show that the compact metric space X is separable, i.e., has a countable dense subset S.

Given a positive integer n and a point $x \in X$, let

$$B(x, 1/n) = \{y \in X : d(x, y) < 1/n\},$$

the open ball of radius $1/n$, centered at x. For a given n, the collection of all these balls as x runs through X is an open cover of x, so (because X is compact) there is a finite subcollection that also covers X. Let S_n denote the collection of centers of the balls in this finite subcollection. Thus S_n is a finite subset of X that is "$1/n$-dense" in the sense that every point of X lies within $1/n$ of a point of S_n. Clearly the union S of all the sets S_n is countable, and dense in X.

STEP II. We find a subsequence of $\{f_n\}$ that converges pointwise on S.

This is a standard diagonal argument. Let's list the (countably many) elements of S as $\{x_1, x_2, \ldots\}$. Then the numerical sequence $\{f_n(x_1)\}_{n=1}^{\infty}$ is bounded, so by Bolzano-Weierstrass it has a convergent subsequence, which we'll write using double subscripts: $\{f_{1,n}(x_1)\}_{n=1}^{\infty}$. Now the numerical sequence $\{f_{1,n}(x_2)\}_{n=1}^{\infty}$ is bounded, so it has a convergent subsequence $\{f_{2,n}(x_2)\}_{n=1}^{\infty}$. Note that the sequence of functions $\{f_{2,n}\}_{n=1}^{\infty}$, since it is a subsequence of $\{f_{1,n}\}_{n=1}^{\infty}$, converges at both x_1 and x_2. Proceeding in this fashion we obtain a countable collection of subsequences of our original sequence:

$$\begin{array}{cccc}
f_{1,1} & f_{1,2} & f_{1,3} & \cdots \\
f_{2,1} & f_{2,2} & f_{2,3} & \cdots \\
f_{3,1} & f_{3,2} & f_{3,3} & \cdots \\
. & . & . & \cdots \\
. & . & . & \cdots \\
. & . & . & \cdots
\end{array}$$

where the sequence in the n-th row converges at the points $x_1, \ldots, x_n$, and each row is a subsequence of the one above it.

Thus the *diagonal sequence* $\{f_{n,n}\}$ is a subsequence of the original sequence $\{f_n\}$ that converges at each point of S.

STEP III. Completion of the proof.

Let $\{g_n\}$ be the diagonal subsequence produced in the previous step, convergent at each point of the dense set S. Let $\varepsilon > 0$ be given, and choose $\delta > 0$ by equicontinuity of the original sequence, so that $d(x, y) < \delta$ implies $|g_n(x) - g_n(y)| < \varepsilon/3$ for each $x, y \in x$ and each positive integer n. Fix $M > 1/\delta$ so that the finite subset $S_M \subset S$ that we produced in Step I is δ-dense in X. Since $\{g_n\}$ converges at each point of S_M, there exists $N > 0$ such that

$$(*) \qquad n, m > N \Rightarrow |g_n(s) - g_m(s)| < \varepsilon/3 \qquad \forall s \in S_M.$$

Fix $x \in X$. Then x lies within δ of some $s \in S_M$, so if $n, m > M$:

$$|g_n(x) - g_m(x)| \le |g_n(x) - g_n(s)| + |g_n(s) - g_m(s)| + |g_m(s) - g_m(x)|$$

The first and last terms on the right are $< \varepsilon/3$ by our choice of δ (which was possible because of the equicontinuity of the original sequence), and the same estimate holds for the middle term by our choice of N in (*). In summary: given $\varepsilon > 0$ we have produced N so that for each $x \in X$,

$$m, n > N \Rightarrow |g_n(x) - g_m(x)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus on X the subsequence $\{g_n\}$ of $\{f_n\}$ is uniformly Cauchy, and therefore uniformly convergent. This completes the proof of the Arzela-Ascoli Theorem. □