Linear Fractional Transformations

The theory of Linear Fractional Transformations (LFT's) in the complex plane is one of the most beautiful and useful tools in complex analysis.

The Schwarzian derivative Sf can be defined for holomorphic maps f (i.e., complex differentiable maps) of the complex plane. It turns out that $Sf \equiv 0$ iff f is a linear fractional transformation.

We recall some of the properties of LFT's.

Def. A linear fractional transformation $T(z)$ of the complex variable z is a map of the form

$$T(z) = \frac{az+b}{cz+d}$$

where a, b, c, d are complex numbers such that $ad - bc \neq 0$.

Let $GL(2, \mathbf{C})$ denote the set of 2×2 complex matrices with non-zerio determinant. Note that we can associate a matrix $A \in GL(2, \mathbf{C})$ with T by the assignment

$$T(z) = \frac{az+b}{cz+d} \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We write this as assignment as the map $T \to A_T$.

Exercise. If A_T and A_S are the matrices associated to the LFT's T and S, respectively, then $A_{T \circ S} = A_T A_S$. That is, the matrix associated to $T \circ S$ is the product of the matrices associated to T and S. Since $GL(2, \mathbf{C})$ is a group under matrix multiplication, it follows that the set of LFT's is a group under composition.

Also, since the inverse A^{-1} of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has the simple formula

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

we can immediately write the inverse of the LFT

$$T(z) = \frac{az+b}{cz+d}.$$

We consider some special LFT's.
Let $z \in \mathbf{C}$.
The map $H(z)$ is

a homothety	if	$H(z) = \alpha z \ \ \exists \alpha \in \mathbf{R}, \alpha \neq 0$
a rotation	if	$H(z) = cz \ \ \exists c \in \mathbf{C}, \mid c \mid = 1$
a translation	if	$H(z) = z + b \ \ \exists b \in \mathbf{C}$
the inversion	if	$H(z) = \frac{1}{z}$

We call these *elementary* LFT's.
Exercises.

1. Every LFT is a composition of elementary ones.

2. A LFT takes takes lines or circles in $\mathbf{C}$ onto lines or circles. (Hint: Do this for the elementary ones first. Then use the fact that every LFT is a composition of elementary ones.)

3. A LFT T is uniquely determined by its image at three distinct points.

Let z_1, z_2, z_3 be three distinct points in $\mathbf{C}$. There is a unique LFT T such that $T(z_1) = 0, T(z_2) = 1$, and $T(z_3) = \infty$.

The formula for T is

$$T(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

We idenitfy the unit sphere S^2 in $\mathbf{R}^3$ with the extended complex plane $\bar{\mathbf{C}} = \mathbf{C} \bigcup \{\infty\}$ using stereographic projection. Here $S^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$.

Let $\mathbf{x}_1 = \Phi(\mathbf{x})$ be the map from S^2 to $\mathbf{C}$ obtained as follows. Let ℓ be a line through $(0, 0, 1)$ which meets the plane $\{(u_1, u_2, u_3) : u_3 = 0\}$. There are unique points $\mathbf{x} = (x_1, x_2, x_3) \in S^2$ and $\mathbf{x}_1 \in \mathbf{R}^2$ which lie on ℓ. Set $\Phi(\mathbf{x}) = \mathbf{x}_1$.

Identifying the point $(x, y) \in \mathbf{R}^2$ with $x + iy \in \mathbf{C}$, there is a simple formula for the map Φ.

If we let $\mathbf{x} = (x_1, x_2, x_3) \in S^2$, (so $x_1^2 + x_2^2 + x_3^2 = 1$,) then

$$\Phi(\mathbf{x}) = \frac{x_1 + ix_2}{1 - x_3}.$$

We leave the verification of this as an exercise (see Ahlfors, Complex Analysis).

Observe that

$$\begin{array}{rcl} \mathbf{x} \text{ in upper hemisphere} & \Rightarrow & \mid \Phi(\mathbf{x}) \mid > 1 \\ \mathbf{x} \text{ in lower hemisphere} & \Rightarrow & \mid \Phi(\mathbf{x}) \mid < 1 \\ \mathbf{x} \text{ in } S^2 \bigcap \mathbf{R}^2 & \Rightarrow & \Phi(\mathbf{x}) = \mathbf{x} \end{array}$$

We think of the point $(0, 0, 1)$ in S^2 as the geometric representation of the *point at infinity* in the extended complex plane $\bar{C}$.

The map Φ (or its inverse) is called *stereographic projection.* It is obvious that Φ^{-1} carries lines in $\mathbf{C}$ to circles through $(0, 0, 1)$. It can also be shown that Φ^{-1} takes circles in $\mathbf{C}$ onto circles on S^2.

Let T be a LFT. Then, T takes lines or circles in $\mathbf{C}$ into other lines or circles. So, the lifted map to S^2 defined by $\Phi^{-1} T \Phi$ takes circles to circles on S^2.

Exercises.

1. Find linear fractional transformations carrying the sets E_1 onto E_2 where
   (a) $E_1 = \{z : \mid z \mid < 1\}$, $E_2 = \{z : Im(z) > 0\}$.
   (b) $E_1 = \{z : \mid z - i \mid < \frac{1}{2}\}$,
   $E_2 = \{z : \mid z - 1 \mid < 4\}$.
2. Consider the subset $\mathcal{U}$ of LFT's T_a of the form

$$T_a(z) = \frac{z - a}{1 - \bar{a}z}$$

   where a is a complex number of norm less than 1, and $\bar{a}$ is its complex conjugate.
   (a) Show that $\mathcal{U}$ is a subgroup of the group of all LFT's.
   (b) Show that each $T_a \in \mathcal{U}$ takes the unit circle
   $\{z : \mid z \mid = 1\}$ onto itself.