1. In [6], Smale introduced some theorems which give much information on the structure of certain diffeomorphisms of a compact manifold. A basic question which arose was whether Axiom A [6, §1.6] was dense on any compact manifold. Subsequently, Abraham and Smale showed that Axiom A(a) was not $C^1$ dense on $T^2 \times S^2$ where $T^2$ is the two torus and $S^2$ is the two sphere [1]. We show in this paper that Axiom A(a) is not $C^2$ dense on $S^2$.

We consider the set of all diffeomorphisms of $S^2$ with the uniform $C^r$ topology, $1 < r < \infty$. Recall from [6] that for $f \in \text{Diff}^r(S^2)$ a point $x \in S^2$ is said to be non-wandering if the following is true. For each neighborhood $U$ of $x$ there is a positive integer $n$ such that $f^n(U) \cap U \neq \emptyset$. The set of nonwandering points will be denoted by $\Omega(f)$. Following Smale, we say that $f$ satisfies Axiom A if (a) $\Omega(f)$ has a hyperbolic structure, and (b) the periodic points of $f$ are dense in $\Omega(f)$. One says that a diffeomorphism $g$ of $S^2$ is topologically conjugate ($\Omega$-conjugate) to $f$ if there is a homeomorphism $h:S^2 \rightarrow S^2 \ (h: \Omega(f) \rightarrow \Omega(g))$ satisfying $gh = hf$. $f$ is called $C^r$ structurally stable ($C^r$-stable) if there is a $C^r$ neighborhood $N$ of $f$ such that any $g \in N$ is topologically conjugate ($\Omega$-conjugate) to $f$. The main result we have is the following.

(1.1) THEOREM. There is an open set $N$ in $\text{Diff}^2(S^2)$ such that if $f \in N$, then $f$ does not satisfy Axiom A(a) and $f$ is not $C^2$ structurally stable.

The basic idea of the proof of this theorem is to modify Smale’s “horseshoe” example [6, §1.5] to produce a diffeomorphism $L$ of $S^2$ such that, for some $x \in \Omega(L)$, the stable manifold $W^s(x, L)$ is tangent to the unstable manifold $W^u(x, L)$ at $x$ (see [6] for definitions). One then shows that this phenomenon is preserved under small $C^2$ perturbations of $L$. From this, nondensity of Axiom A(a) is immediate, and, with some slight argument, nondensity of structural stability also follows.

We should make several remarks. First, in [1], Abraham and Smale also prove that $\Omega$-stable diffeomorphisms are not $C^1$ dense on $T^2 \times S^2$. Second, Williams has used the “DA” examples of Smale to show that structurally stable diffeomorphisms are not $C^1$ dense on $T^2$ [7]. Third, C. Pugh has shown that Axiom A(b) is $C^1$ dense on all compact manifolds [4]. Finally, the main references for this paper are the papers of Smale, [5] and [6]. In fact, I would suggest that the reader be reasonably familiar with §§1.5 and 1.6 of [6] before proceeding.

In §2, we construct the diffeomorphism $L$ as a natural extension of a diffeomorphism $L$ of the plane. §3 contains some results about Cantor sets which will be
needed. §4 is largely motivation for §5 where the main results about $C^2$ perturbations of $L$ are proved. In §6 we prove that all sufficiently small $C^2$ perturbations of $L$ do not satisfy Axiom A(a), and in §7 we sketch a proof that these small perturbations of $L$ are not $C^2$ structurally stable. We conclude §7 with some remarks about the $\Omega$-instability of these small perturbations of $L$.

I wish to express my thanks to the many mathematicians with whom I discussed this paper. Particular thanks go to M. Hirsch, C. Pugh, and S. Smale for much encouragement and many valuable conversations, and to R. Williams, who read a preliminary version of the paper and made many valuable suggestions.

2. To prove Theorem (1.1), we shall define a $C^\infty$ diffeomorphism $\bar{L}$ of $S^2$ such that there is a $C^2$ neighborhood $N$ of $L$ such that no $f \in N$ satisfies Axiom A(a). We first construct a diffeomorphism $L$ of the plane $\mathbb{R}^2$ so that $L(x) = x$ outside some compact subset $H$ of $\mathbb{R}^2$. Then $L$ induces the diffeomorphism $\bar{L}$ of $S^2$ as follows. Let $\phi: \mathbb{R}^2 \to U$ be a $C^\infty$ diffeomorphism where $U$ is a coordinate patch on $S^2$. Let $\bar{L}(x) = \phi \circ L \circ \phi^{-1}(x)$ for $x \in U$ and $\bar{L}(x) = x$ for $x \notin U$.

We now construct $L$.

Consider the square $Q = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq 1\}$. Let $L: Q \to \mathbb{R}^2$ be such that

(2.1) $L(Q) \cap Q$ has two components $A_1 \subset (x_1 < 0)$ and $A_2 \subset (x_1 > 0)$.

![Figure 2.1](image-url)

(2.2) $L(AB) \cup L(CD) \subset (x_2 < 0)$.

\[
L_\star|_{L^{-1}(A_2)} = \begin{pmatrix} \chi & 0 \\ 0 & 1/\chi \end{pmatrix} \quad \chi < 1/2,
\]

(2.3) $L_\star|_{L^{-1}(A_2 \cup EF \cup B)} = \begin{pmatrix} -\chi & 0 \\ 0 & -1/\chi \end{pmatrix}$ where $HA' = L(A)$, $B' = L(B)$, etc., and $EF = A_2 \cap CD$. 

(2.4) $l \pi_1(A_1) = l \pi_1(A_2) > \text{dist}(A_1, A_2) > 2l \pi_1(L(A_1) \cap A_1)$ where $l$ means horizontal length and $\pi_1 : \mathbb{R}^2 \to (x_2 = 0)$ is the vertical projection.

(2.5) $\text{dist}(A_2, BD) = \text{dist}(A_1, AC) = \beta$ where $\beta$ is small.

For instance, suppose $4\alpha \beta + \text{dist}(A_1, A_2) < l \pi(A_1) - (\beta + 2\alpha \beta)$. Suppose also that estimates analogous to those in (2.4) and (2.5) are valid for $L^{-1}(A_1)$ and $L^{-1}(A_2)$.

Define $L$ on $EFA'B'$ such that there is a subrectangle $Q_1$ with sides parallel to the coordinate axes such that

(2.6) $L(Q_1)$ is fibered by concentric semicircles which are the images of the vertical line segments in $Q_1$.

(2.7) The images of the top and bottom sides of $Q_1$ are on the same horizontal line.

(2.8) $(L|_{Q_1})_*(E_1)$ is perpendicular to $(L|_{Q_1})_*(E_2)$ where $E_1$ is the horizontal tangent space and $E_2$ is the vertical tangent space. We identify these with $(x_2 = 0)$ and $(x_1 = 0)$, respectively.

(2.9) The image under $L$ of the horizontal line segment through the midpoint of $Q_1$ connecting both vertical sides of $Q_1$ is vertical. Call this image $\rho_0$.

\[\text{Figure 2.2}\]
Let $Q = W^s_{-1}(L) = W^u_{-1}(L)$ and, for $i > -1$, let $W^s_i(L) = L^{-1}(W^s_{i-1}(L) \cap L(Q))$ and $W^u_i(L) = L(W^u_{i-1}(L)) \cap Q$.

Let $Q_2$ be a small rectangle bounded on the left by part of the boundary of $A_1$, bounded on the right by part of the boundary of $A_2$, and such that $L(Q_1) \subset \text{interior } Q_2$.

Define $W^s_i(L) = L^2(W^s_{i-2}(L)) \cap Q_2$, $i \geq 2$, and $W^u_i(L) = W^u_i(L) \cap Q_2$, $i \geq 2$.

Let $\tilde{W}^s(L) = \bigcap_{i \geq 2} W^s_i(L)$ and $\tilde{W}^u(L) = \bigcap_{i \geq 2} W^u_i(L)$.

We also assume that (2.10) $\tilde{W}^s_2(L) \cap \rho_0 = \tilde{W}^u_2(L) \cap \rho_0$.

The effect of conditions (2.4), (2.5), and (2.10) is to insure that the sets $W^s_i(L) \cap \rho_0$ and $W^u_i(L) \cap \rho_0$ are $k$-thick Cantor sets in the sense of definition (3.6), with $k > 1$.

This is necessary for the preservation of tangencies between stable and unstable manifolds of small perturbations of $L$, as will become clear in the sequel.

Observe that $L|Q$ is the same as in Smale’s original horseshoe example, so any parts of the stable and unstable manifolds of $L$ which depend only on $L|Q$ are the same as in his example. This is true, in particular, for the set $\tilde{W}^s(L)$ defined above.

Now extend $L$ to be a diffeomorphism of the plane so that outside some large compact set $H$, $L$ is the identity. Assume that all of the picture in Figure 2.2 is contained in the interior of $H$. Then, letting $\bar{L} \in \text{Diff}^\omega(S^2)$ be as above, we proceed to show that for some $C^2$ neighborhood $N$ of $L$, any $f \in N$ has a nonwandering point $x$ whose stable and unstable manifolds are tangent at $x$.

3. We need some results about Cantor sets.

(3.1) **Definition.** By a two component Cantor set $F$ in the interval $I \subset R^1$, we mean one obtained as the intersection of a decreasing sequence of closed sets $F_i \subset I$, $i \geq 2$, where (a) $F_2$ has two components, and (b) for $i \geq 2$, each component of $F_i$ contains precisely two components of $F_{i+1}$. We call such a sequence a defining sequence for $F$. If $\{F_i\}_{i \geq 2}$ is a defining sequence for the two component Cantor set $F$, the two-gap, $g_2$, of $F$ or $F_i$ is the component of $I - F_2$ which is between the two components of $F_2$. An i-gap, $g_i$, is a component of $I - F_i$ between two components of $F_i$ such that $g_i \subset F_{i-1}$. Let $F$ be a two component Cantor set in $I$, $g_i$ be an i-gap, and $c_i$ be the smaller of the two components of $F_i$ adjacent to
Let $lg_i = sup g_i - inf g_i$, and $lc_i = sup c_i - inf c_i$. Let $k > 0$. Say that a two component Cantor set $F \subset I$ is a $k$-Cantor set in $I$ if there is a defining sequence $\{F_i\}_{i \geq 2}$ for $F$ satisfying

(3.2) $sup F_2 = sup I$, $inf F_2 = inf I$.

(3.3) For any $i$-gap $g_i$ of $F_i$, $lc_i/lg_i > k$, $i \geq 2$.

(3.4) If $c$ is a component of $F_i$, $i \geq 2$, then $c - F_{i+1}$ is an open interval in $c$.

Thus a $k$-Cantor set is one which has a defining sequence $\{F_i\}_{i \geq 2}$ such that $F_{i+1}$ is obtained by removing open intervals from the components of $F_i$ in such a way that (3.3) is satisfied. For instance, the standard Cantor middle third set is a $k$-Cantor set in the unit interval for any $0 < k < 1$. When we speak of a defining sequence for a $k$-Cantor set, we mean one which satisfies conditions (3.2), (3.3), and (3.4).

(3.5) **Lemma.** If $F$ is a $k$-Cantor set in $I$, $G$ is a $k$-Cantor set in $J$, $k > 1$, $I \cap J \neq \emptyset$, and neither $F$ nor $G$ is completely contained in a gap of the other, then $F \cap G \neq \emptyset$.

**Proof.** Let $\{F_i\}_{i \geq 2}$ and $\{G_i\}_{i \geq 2}$ be defining sequences for $F$ and $G$, respectively. If $F \cap G = \emptyset$, then there is an $m \geq 2$ such that $F_m \cap G_m = \emptyset$. Let $g$ be the smallest gap of $F_m$ or $G_m$ which contains a component of the other. Suppose for definiteness $g$ is a gap of $F_m$, and $g$ contains the component $c$ of $G_m$. In the interval $I \cap J$, either there is a gap $g_r$ of $G_m$ to the right of $c$ such that $lg_r > lg$ or max $G_m \in g$. If $g_r$ exists, assume it to be the first such gap; if not, let $g_r = max G_m$. Similarly, either there is a first gap $g_l$ of $G_m$ to the left of $c$ such that $lg_l > lg$ or min $G_m \in g$. Adjust the definition of $g_r$ accordingly. It follows from the hypothesis of the lemma that at least one of $g_l$ and $g_r$ exists. If they both exist, let $min \{lg_l, lg_r\}$ be the smaller of the two lengths; if one does not exist, let $min \{lg_r, lg_l\}$ be the length of the existing gap. Let $c_1$ be the interval from the right endpoint of $g_l$ to the left endpoint of $g_r$. Now it follows from the choices of $g, g_l, g_r$, and $c_1$ that $c_1 < g$ so that $lc_1 < lg$. Further, if $i \leq m$ is the first integer such that both gaps $g_l$ and $g_r$ appear in $G_i$, then there are a $c_1$ and $g_i$ for $G_i$ as defined above, satisfying $lc_1 \leq lc_i$ and $min \{lg_l, lg_r\} \leq lg_i$. In fact, $g_i$ can be chosen as either $g_l$ or $g_r$, and $c_1$ will then be its adjacent component which is contained in $c_1$. Thus

$$lc_1 \leq lc_i < lg \leq min \{lg_l, lg_r\} \leq lg_i < lc_i/k$$

which is a contradiction.

We will need to generalize the notion of $k$-Cantor set to the situation in which the endpoints of the components of the defining sequence are not in the Cantor set.

(3.6) **Definition.** Let $F \subset I$ be a two component Cantor set with defining sequence $\{F_i\}$. Let $g_i, c_i$ be as above. For $m \geq i$, let $g_{im}$ be the component of $I - F_m$ which contains $g_i$. Let $c_{fi}$ be the union of the components of $F_m$ which are contained in $c_i$. Let $lc_{fi} = sup c_{fi} - inf c_{fi}$ and $lg_{fi} = sup g_{im} - inf g_{im}$. Let $k, k_1, k_2 > 0$. We say that $F$ is $(k_1, k_2)$-thick if there is a defining sequence $\{F_i\}$ for $F$ such that for all $i$-gaps, $g_i, i \geq 2$, and all $m \geq i, k_1 < lc_{fi}/lg_{fi} < k_2$. We say that $F$ is $k$-thick if there is a defining sequence $\{F_i\}$ for $F$ such that for all $i$-gaps, $i \geq 2$, 3.19
and all $m \geq i, k < l_{cm}/l_{gm}$. Then we have the following lemma, the proof of which is similar to that of Lemma (3.5).

(3.7) Lemma. Let $F$ and $G$ be two $k$-thick Cantor sets in $I$. Suppose $k > 1$, neither $F$ nor $G$ is completely contained in a gap of the other, and $\max(\min F, \min G) < \min(\max F, \max G)$. Then $F \cap G \neq \emptyset$.

4. To motivate the use of thick Cantor sets in establishing the stability of the tangency condition in §2, and to motivate the proof of Lemma (5.1), we consider a perturbation problem in a one-dimensional setting. The result of this section will not be used in the sequel, so the reader who wishes to skip the section may do so.

Let $V$ be a closed bounded interval contained in the reals $R^1$. We assume that all functions which we consider in this section are defined on $V$ and map $V$ into itself. All closed intervals $I, J$, etc., which we consider are assumed to be contained in the interior of $V$.

Let $I \subseteq V$ be a closed interval, and let $f_1$ and $f_2$ be two contracting (derivative everywhere between 0 and 1) $C^r$ diffeomorphisms which map $I$ into itself such that $f_1(I) \cap f_2(I) = \emptyset$. Let $F(f_1, f_2) = I$ and, for $i \geq 2$, $F(f_1, f_2) = f_i(F_{i-1}(f_1, f_2)) \cup f_2(F_{i-1}(f_1, f_2))$. Then $F(f_1, f_2) \equiv \bigcap_{i \geq 2} F(f_1, f_2)$ is a two component Cantor set in $I$. We say that $F(f_1, f_2)$ is defined on $I$. By a $C^r$ perturbation or approximation $F(h_1, h_2)$ of $F(f_1, f_2)$ we mean a Cantor set defined as above on an interval $J \subseteq V$ where

(4.1) the endpoints of $J$ are close to those of $I$, and
(4.2) $h_1$ and $h_2$ are $C^r$ close on $V$ to $f_1$ and $f_2$, respectively.

Now we ask the following question.

(4.3) Given $F(f_1, f_2)$, can one find perturbations $F(h_1, h_2)$ and $F(h_1, h_2)$ arbitrarily close to $F(f_1, f_2)$ such that $F(h_1, h_2) \cap F(h_1, h_2) = \emptyset$?

We do not intend to discuss this question in detail, but rather to consider only those aspects of it which relate to the diffeomorphism $L$ of §2. In this connection we have the following proposition.

(4.4) Proposition. Let $k > k' > 1$. Assume that $F(f_1, f_2)$ is defined as above, that $F(f_1, f_2)$ is a $k$-Cantor set in $I$, and that $f_1$ and $f_2$ are linear contracting diffeomorphisms of $I$ into itself, i.e. the second derivatives $f_1''(x)$ and $f_2''(x)$ are identically zero on $I$. Then any $F(h_1, h_2)$ which is sufficiently $C^2$ close to $F(f_1, f_2)$ is a $k'$-Cantor set on its interval of definition.

We observe that combining this proposition and Lemma (3.1), we obtain that, in general, the answer to question (4.3) is no.

Proof. Let $c = \frac{1}{2} \inf \{ \min(f_i'(x), f_j'(x)) : x \in I \}$. Let $g(h)$ be an $i$-gap of any approximation $F(h_1, h_2)$ of $F(f_1, f_2)$, and $c_j(h)$ be its adjacent component. Clearly, if $h_1$ and $h_2$ are close enough to $f_1$ and $f_2$, then $l_{g_2(h)}/l_{c_2(h)} < 1/k < 1/k'$.

Fix $i > 2$. Then there is a sequence $n_3, n_4, ..., n_i$ where each $n_j = 1$ or $2$, $3 \leq j \leq i$, such that

$h_{n_i} \circ h_{n_{i-1}} \circ ... \circ h_{n_3}(g_2(h)) = g(h)

and

$h_{n_i} \circ h_{n_{i-1}} \circ ... \circ h_{n_3}(c_2(h)) = c(h)$.\n


For $3 \leq j \leq i$, let $g_j'(h) = h_{n_0} \circ \ldots \circ h_{n_3}(g_3(h))$ and $c_j(h) = h_{n_0} \circ \ldots \circ h_{n_3}(c_3(h))$.

Let $\alpha_j = lg_j'(h)/g_j'(h)$ and $\beta_j = lc_j'(h)/c_j'(h)$.

Then $lg_j(h) = \alpha_j \ldots \alpha_{j-1} lg_j'(h)$ and $lc_j(h) = \beta_j \ldots \beta_{j-1} lc_j'(h)$.

Let $m_j = l(g_{j-1}(h) \cup c_{j-1}(h))$.

Now, $\alpha_j$ is the derivative of $h_j$ at some point of $g_{j-1}$, and $\beta_j$ is the derivative of $h_j$ at some point of $c_{j-1}$. If $\varepsilon > 0$ is given, and $h_1$ and $h_2$ are $C^2$ close to $f_1$ and $f_2$, then the mean value theorem yields (since $f_1$ and $f_2$ are linear)

$$ \frac{\alpha_j - \beta_j}{m_j} < \varepsilon. $$

Thus, for $3 \leq j \leq i$,

$$ x_j/\beta_j < (\beta_j + m_j \varepsilon)/\beta_j. $$

Thus,

$$ \frac{lg_j(h)}{lc_j(h)} < \left(\prod_{j=3}^{i} 1 + \frac{m_j \varepsilon}{c} \right) \frac{lg_2(h)}{lc_2(h)}. $$

Now one can easily check that $\sum_{j=3}^{i} m_j < \infty$. Hence, choosing $\varepsilon$ appropriately, the proposition follows.

5. In what follows, all of our approximations will be with respect to the $C^2$ metric $d$ on $\text{Diff}^2(H)$. We shall apply the results of §3 to obtain some results about diffeomorphisms $C^2$ near $L$.

Recall $\overline{W}^s(L) = \bigcap_{i \geq 2} \overline{W}^s_i(L)$ and $\overline{W}^u(L) = \bigcap_{i \geq 1} \overline{W}^u_i(L)$. Then, if $\gamma$ is a $C^1$ compact arc in $Q_2$ which is $C^1$ near $\rho_0$, $\overline{W}^s(L) \cap \gamma$ and $\overline{W}^u(L) \cap \gamma$ are $(k_1, k_2)$-thick Cantor sets for some $1 < k_1 < k_2$. The next lemma asserts that this is true for a perturbation of $L$. Note that for $f$ close to $L$ we may define $\overline{W}^s_i(f), \overline{W}^u_i(f), \overline{W}^s(f)$ and $\overline{W}^u(f)$ as we did for $L$. We observe that if $x \in \overline{W}^s(f) \cap \overline{W}^u(f), x \in \Omega(f)$, since it is an accumulation point of homoclinic points of a fixed point of $f$ (see [6] for definitions). For $f$ close to $L$, $\gamma$ $C^1$ near $\rho_0$, let $\overline{W}^s_{i_1}(f) = \overline{W}^s_i(L) \cap \gamma$. Let $g^*_{i_1}(f)$ be an $i$-gap of $\overline{W}^s_{i_1}(f), c^*_{i_1}(f)$ be an adjacent component to $g^*_{i_1}(f)$. Make similar definitions for $g^*_{i_2}(f), c^*_{i_2}(f)$, $\overline{W}^u_{i_1}(f), \overline{W}^u_i(f)$, etc. For $\gamma$ $C^1$ near $\rho_0$, let $|\gamma - \rho_0|$ denote the $C^1$ distance between $\gamma$ and $\rho_0$.

(5.1) Lemma. Let $1 < k_1 < k_2$ be such that $\overline{W}^s_{\rho_0}(L)$ and $\overline{W}^u_{\rho_0}(L)$ are $(k_1, k_2)$-thick Cantor sets. Let $1 < k'_1 < k_1 < k_2 < k'_2$. Then there is an $\alpha > 0$ and a $C^2$ neighborhood $N$ of $L$ such that for any compact arc $\gamma$ in $Q_2$ with $|\gamma - \rho_0| < \alpha$, and any $f \in N, \overline{W}^s_i(f)$ and $\overline{W}^u_i(f)$ are $(k_1', k_2')$-thick Cantor sets.

Proof. We first observe that by $C^1$ dependence of the stable and unstable manifolds on $f$ near $L$ (Smale [5]), it is sufficient to prove that for $f$ $C^2$ near $L$, $\overline{W}^s_{\rho_0}(f)$ and $\overline{W}^u_{\rho_0}(f)$ are $(k_1', k_2')$-thick Cantor sets. We prove this for $\overline{W}^s_{\rho_0}(f)$. The proof for $\overline{W}^u_{\rho_0}(f)$ is similar to that for $\overline{W}^s_{\rho_0}(f)$. One does the estimates on $f^{-1}(\overline{W}^s_{\rho_0}(f))$ and then carries them over to $\overline{W}^s_{\rho_0}(f)$.

Let $c = \frac{1}{2} \inf \{ |D(L^{-1}x)v| : v \text{ is a unit vector in } E_1 \times E_2, x \in H \}.$

We first prove that for $i > 2$, $f$ close to $L$, $lg_{i}(f)/lc_{i}(f) < 1/k'_i$, where $g^*_{i}(f)$ is an $i$-gap of $\overline{W}^s_{\rho_0}(f)$ and $c^*_{i}(f)$ is an adjacent component. We have that

$$ lg_{2}(L)/lc_{2}(L) < 1/k_1 < 1/k'_1. $$

$$ \overline{W}^s(L) = \bigcap_{i \geq 2} \overline{W}^s_i(L) $$

$$ \overline{W}^u(L) = \bigcap_{i \geq 1} \overline{W}^u_i(L) $$

$$ \overline{W}^s(f) $$

$$ \overline{W}^u(f) $$

$$ \overline{W}^s_{i_1}(f) $$

$$ \overline{W}^u_{i_1}(f) $$

$$ \gamma $$

$$ |\gamma - \rho_0| $$

$$ C^1 $$

$$ N $$

$$ \overline{W}^s_{\rho_0}(L) $$

$$ \overline{W}^u_{\rho_0}(L) $$

$$ k_1 $$

$$ k_2 $$

$$ k'_1 $$

$$ k_1' $$

$$ k_2' $$

$$ \alpha $$

$$ C^2 $$

$$ f $$

$$ \rho_0 $$

$$ f^{-1}(\overline{W}^s_{\rho_0}(f)) $$

$$ \overline{W}^s_{\rho_0}(f) $$

$$ \overline{W}^u_{\rho_0}(f) $$

$$ g^*_{i}(f) $$

$$ c^*_{i}(f) $$

$$ i $$

$$ k'_i $$

$$ \overline{W}^s_{\rho_0}(f) $$

$$ \overline{W}^u_{\rho_0}(f) $$

$$ f^{-1}(\overline{W}^s_{\rho_0}(f)) $$

$$ \overline{W}^s_{\rho_0}(f) $$

$$ g^*_{i}(f) $$

$$ c^*_{i}(f) $$

$$ i $$

$$ k'_i $$
Fix \( i > 2 \). Define \( g^i(f) = g^i(f) \), \( c^i(f) = c^i(f) \), \( g^j(f) = f(g^j(f)) \), and 
\( c^j(f) = f(g^j(f)) \) for \( 3 \leq j \leq i \).

Let \( \varepsilon_1 \) be such that if \( d(f, L) < \varepsilon_1 \), then 
\[
lc^2(f)/lc^2(f) < 1/k_1 < 1/k_1.
\]

This can be done since, by taking \( \varepsilon_1 \) small, independent of \( i \), if \( d(f, L) < \varepsilon_1 \) the arcs \( g^2(f) \) and \( c^2(f) \) are nearly vertical \([5]\), and thus \( lc^2(f)/lc^2(f) \) is close to \( lc^2(f)/lc^2(f) \).

Let \( \varepsilon_2 \) be such that 
\[
\prod_{i=0}^{\infty} \left( 1 + \frac{\varepsilon_2}{2e} \right) \frac{1}{k_1} < \frac{1}{k_1}.
\]

Define \( \alpha_j = lc^2(f)/lc^2(f) \) and \( \beta_j = lc^2(f)/lc^2(f) \) for \( 3 \leq j \leq i \). Let 
\( m(f) = l(g^j(f) \cup c^j(f)) \). Thus, 
\[
lc^2(f) = \alpha_1 \alpha_{i-1} \ldots \alpha_{3} lc^2(f)
\]

and 
\[
lc^2(f) = \beta_1 \beta_{i-1} \ldots \beta_{3} lc^2(f).
\]

Now \( \alpha_j \) may be thought of as the derivative of \( f^1_{g^j(f)} c^j(f) \) at some point of 
\( c^j(f) \), and \( \beta_j \) may be thought of as the derivative of \( f^1_{g^j(f)} c^j(f) \) at some point of 
\( c^j(f) \). Since \( D^2L^1 = 0 \) on \( A_1 \), for \( \varepsilon_1 \) smaller, if necessary, \( |D^2f^1| < \varepsilon_2 \) on 
\( A_1 \). Since the arcs are nearly vertical, we have, with the proper orientations, if 
\( \alpha_j > \beta_j \), then \( (\alpha_j - \beta_j)/m(f) < \varepsilon_2 \) by the one-dimensional mean value theorem.

Thus for \( \alpha_j > \beta_j \), 
\[
\frac{\alpha_j}{\beta_j} < 1 + \frac{m_{e_2}}{\beta_j}.
\]

But again for \( \varepsilon_1 \) small, \( \beta_j > c \) and \( m_j < \frac{1}{3} m_{j-1} \) for \( 3 \leq j \leq i \).

So \( \alpha_j/\beta_j < 1 + \frac{m_{e_2}/c}{\beta_j} \), for \( 3 \leq j \leq i \), and \( m_j < (1/2^{j-2})m_2 < 1/2^{j-3} \). Thus, 
\[
lc^2(f)/lc^2(f) < \prod_{j=3}^{i} \left( 1 + \frac{m_{e_2}}{c} \right) \
\]

To prove that for \( m \geq i \), \( lc^2(f)/lc^2(f) < 1/k_1 \) for \( f \) close to \( L \), we need only 
to make estimates of the first derivative of \( f \) at certain points. Note that by the 
above argument, we may prove that \( lc^2(f)/lc^2(f) < \lambda \) where \( 1/k_1 < \lambda < 1/k_1 \) for 
\( f \) close to \( L \). We observe that for \( f \) close to \( L \), \( \sup g^i(f) - \inf g^i(f) \) and 
\( \inf g^i(f) - \sup g^i(f) \) are small enough so that \( lc^2(f) \) is close to \( lc^2(f) \). Similarly, 
\( lc^2(f) \) is close to \( lc^2(f) \). Making these estimates refined enough we may prove 
\( lc^2(f)/lc^2(f) < 1/k_1 \).

Now notice that we may similarly prove that for \( f \) close to \( L \), 
\( lc^2(f)/lc^2(f) < k_2 \) for all \( i \) and all \( m \geq i \). This proves Lemma (5.1).
Our goal is to prove that for \( f C^2 \) near \( L \), \( \bar{W}^u(f) \) and \( \bar{W}^s(f) \) have a point of tangency. Using Lemmas (5.1) and (3.2), we can show that for \( \gamma C^1 \) near \( \rho_0 \) and \( f C^2 \) near \( L \), \( \bar{W}^u(f) \cap \bar{W}^s(f) \neq \emptyset \). Let us orient the arcs of \( \bar{W}^u(L) \) and \( \bar{W}^s(L) \) so that for \( \gamma \) on the left of \( \rho_0 \), the angles between intersecting arcs of \( \bar{W}^u(L) \) and \( \bar{W}^s(L) \) are all less than zero. Then for \( \gamma \) on the right of \( \rho_0 \), the corresponding angles of intersection will all be greater than zero. It is clear that for \( f C^1 \) close to \( L \), an analogous result is true about angles of intersection of \( \bar{W}^u(f) \) and \( \bar{W}^s(f) \) for appropriate \( \gamma \).

This makes it plausible that the desired points of tangency should exist. However, to prove they actually do exist, we need to know that, for any \( i \geq 2 \), each component of \( f^{-i}(\bar{W}^u(f)) \) has a nearly constant horizontal width. This is essentially the content of Lemma (5.3). To prove this lemma we will need the following theorem.

(5.2) Theorem (M. Hirsch and C. Pugh [2]). Let \( f \) be a \( C^2 \) Anosov diffeomorphism of the two torus \( T^2 \). Let the tangent bundle have the continuous hyperbolic splitting \( T(T^2) = E^s \oplus E^u \) where \( \| (Df)E^s \| < 1 \) and \( \| (Df)E^u \|^{-1} < 1 \). Then \( E^s \) and \( E^u \) are \( C^1 \) subbundles of \( T(T^2) \), and if \( f_1 \) is \( C^2 \) close to \( f \), the unit ball bundles of the invariant subbundles \( E^s_1 \) and \( E^u_1 \) are \( C^1 \) close to those of \( E^s \) and \( E^u \), respectively.

I should remark that the statement about \( C^1 \) dependence in the above theorem is not actually written down in [2]. However, both M. Hirsch and C. Pugh informed me that it follows from their methods.

I am indebted to C. Pugh for telling me about the above theorem and for a conversation which was very helpful for the proof of the following lemma.

Notice that the definitions of \( \bar{W}^u(f) \), \( \bar{W}^s(f) \), etc. make sense for any diffeomorphism \( \bar{Q} \) of \( \bar{Q} \) which is \( C^1 \) close enough to \( \bar{Q} \) and any \( f C^1 \) close enough to \( L \). That is, let \( \bar{W}^u(f) = f^{-1}(\bar{W}^u(f) \cap f(\bar{Q})) \), etc. We shall call any such \( \bar{Q} \) near \( \bar{Q} \) a “square” near \( \bar{Q} \).

(5.3) Lemma. Let \( 0 < \delta_1 < 1 < \delta_2 \). Then there exists a neighborhood \( N \) of \( L \) and a \( C^2 \) “square” \( \bar{Q} \) near \( \bar{Q} \) such that the following is true. Let \( f \in N, i \geq 2, \gamma_1 \) and \( \gamma_2 \) be \( C^1 \) curves nearly horizontal in \( \bar{Q} \), and define \( \gamma_1^f \) and \( \gamma_2^f \) to be components of \( \bar{W}^u(f) \cap \gamma_1 \) and \( \bar{W}^u(f) \cap \gamma_2 \) contained in the same strip of \( \bar{W}^u(f) \). Then

\[
\delta_1 < \frac{\lambda^{\gamma_1^f}(f)}{\lambda^{\gamma_2^f}(f)} < \delta_2.
\]

Proof. We may assume \( \gamma_1 \) and \( \gamma_2 \) are horizontal in \( \bar{Q} \). Let \( f_0 \) be an Anosov diffeomorphism of the two torus \( T^2 \) (see Smale [6]) such that

(5.4) There is a subset \( \bar{Q}' \) of \( T^2 \) such that \( f_0(\bar{Q}') \cap \bar{Q}' \) has two components \( A_1' \) and \( A_2' \).

(5.5) There is a diffeomorphism \( d_1 : \bar{Q} \rightarrow \bar{Q}' \) such that \( d_1(A_1) = A_1' \) and \( d_1(A_2) = A_2' \), \( d_1(L^{-1}(A_1)) = f_0^{-1}(A_1') \), and \( d_1(L^{-1}(A_2)) = f_0^{-1}(A_2') \).

Let \( e \) be the isometry \( A_2 \rightarrow A_2 \) which is rotation by \( \pi \) about the midpoint of \( A_2 \).

(5.6) There are disjoint neighborhoods \( U_1 \) of \( A_1 \) and \( U_2 \) of \( A_2 \) and a diffeomorphism \( d_2 : U_1 \cup U_2 \rightarrow T^2 \) such that \( d_2|_{U_1} = d_1 \) and \( d_2|_{U_2} = d_1 \circ e \).
(5.7) \( d_1(AC) \) and \( d_1(BD) \) are segments of unstable manifolds of \( f_0 \), and \( d_1(AB) \) and \( d_1(CD) \) are segments of stable manifolds of \( f_0 \).

(5.8) \( f_0 = d_2 \circ f \circ d_1^{-1} \) on a neighborhood of \( f_0^{-1}(A'_1 \cup A'_2) \).

It is easy to see that such an Anosov diffeomorphism exists. Now let \( f \) be a slight \( C^2 \) perturbation of \( L \). Then for some neighborhood \( U_3 \) of \( f_0^{-1}(A'_1 \cup A'_2) \), \( d_2 \circ f \circ d_1^{-1} \big|_{U_3} \) is a slight \( C^2 \) perturbation of \( f_0 \big|_{U_3} \). Let \( f'_0 \) be an Anosov diffeomorphism of \( T^2 \) which is close to \( f_0 \) such that \( f'_0 = d_2 \circ f \circ d_1^{-1} \) on a neighborhood of \( f_0^{-1}(A'_1 \cup A'_2) \) and \( f'_0 = f_0 \) on \( T^2 - U_3 \). Then let \( \hat{Q} \) be a new "square" in \( T^2 \), near \( Q' \), such that \( f'_0 \big|_{\hat{Q}} \) is close to \( f_0 \big|_{\hat{Q}} \) and \( \hat{Q} \) is bounded by segments of the stable and unstable manifolds of \( f_0 \). Now if \( f \) is \( C^2 \) close to \( L \), \( f'_0 \) will be \( C^2 \) close to \( f_0 \). Hence Theorem 3.2 gives that, for \( i \geq 2 \),

\[
\frac{\|c^{u}_{l_1}(f'_0)\|}{\|c^{u}_{l_1}(f_0)\|} \text{ is close to } \frac{\|c^{u}_{l_1}(f'_0)\|}{\|c^{u}_{l_1}(f_0)\|}.
\]

Here we assume \( c^{u}_{l_1}(f'_0) \) and \( c^{u}_{l_1}(f_0) \) are defined with respect to \( \hat{Q} \), and \( c^{u}_{l_1}(f'_0) \) and \( c^{u}_{l_1}(f_0) \) are defined with respect to \( \hat{Q} \). Now let \( \hat{Q} = d_1^{-1}(\hat{Q}) \) and the lemma follows.

6. Now we conclude the proof of the nondensity of Axiom A(a) by showing that for \( f \) \( C^2 \) close enough to \( L \), \( \hat{W}^s(f) \) and \( \hat{W}^u(f) \) have a point of tangency. Recall that such a point will belong to \( \Omega(f) \) since it will be an accumulation point of homoclinic points of a fixed point of \( f \).

In this section we will make the simplifying assumption that for all perturbations \( f \) of \( L \) which we consider, \( \hat{W}^s(f) = \hat{W}^s(L) \) for \( n \geq 2 \). This avoids technical difficulties and indicates the main ideas needed in the proof. To make the proof rigorous we would have to enlarge each \( \hat{W}^s(f) \) to obtain a \( C^2 \) foliation of \( Q_2 \) and proceed as below with respect to these foliations. Since, for \( f \) \( C^1 \) close to \( L \), all of these foliations can be made uniformly \( C^1 \) close to the natural horizontal foliation of \( Q_2 \), we are justified in making the estimates with respect to this foliation.

Let \( f \) be close to \( L \) and let \( n \geq 2 \). Note that \( \hat{W}^u(f) \) has \( 2^n - 1 \) components and thus the boundary of \( \hat{W}^u(f) \) consists of \( 2^n \) curves. Label these curves \( \xi^1_n(f), \xi^2_n(f), \ldots, \xi^{2^n}_n(f) \) so that \( \xi^i_n(f) \) is below \( \xi^{i+1}_n(f) \), \( \xi^{i+1}_n(f) \) is below \( \xi^i_n(f) \), etc. For each \( i = 1, 2, \ldots, 2^n \), let \( \psi^i_n(f) \) be a point of \( \xi^i_n(f) \) at which \( \xi^i_n(f) \) assumes its maximum with respect to the horizontal foliation of \( Q_2 \). Assume that \( f \) is \( C^1 \) close enough to \( L \) so that for each \( i \), \( \psi_i^i_n(f) \) is in the interior of the curve \( \xi^i_n(f) \), i.e. \( \psi^i_n(f) \) is not an endpoint of \( \xi^i_n(f) \). Note that \( \xi^i_n(f) \) is tangent to the horizontal foliation of \( Q_2 \) at \( \psi^i_n(f) \). For \( i = 1, 3, 5, \ldots, 2^n - 1 \), let \( c^i_n(f) \) be the closed rectangular strip in \( Q_2 \)
whose boundary consists of the horizontal line segment in $Q_2$ through $\psi_n(f)$, the horizontal line segment in $Q_2$ through $\psi^+_n(f)$ and parts of the vertical edges of $Q_2$. Then let $F_n(f) = \bigcup_{i=1,3,...,2^n-1} c_n(f)$. Note that if $f$ is close to $L$, $\bigcap_{n \geq 2} F_n(f) \equiv F(f)$ is the product of a horizontal line segment in $Q_2$ and a Cantor set. Let $F_{\rho_0}(f) = F(f) \cap \rho_0$.

(6.1) Lemma. There is a $C^2$ neighborhood $N$ of $L$ such that if $f \in N$, then $F_{\rho_0}(f)$ is a $k$-thick Cantor set where $k > 1$.

Proof. This follows from Lemma (5.1), Lemma (5.3), and the construction of $F_{\rho_0}(f)$.

To conclude the proof of the first part of Theorem (1.1) we see that for some small $C^2$ neighborhood $N$ of $L$, if $f \in N$, then $F_{\rho_0}(f)$ and $\overline{W}_{\rho_0}(f)$ are $k$-thick Cantor sets for some $k > 1$. By restricting $N$ further if necessary, we may assume the hypotheses of Lemma (3.7) are satisfied by $F_{\rho_0}(f)$ and $\overline{W}_{\rho_0}(f)$. Thus $F_{\rho_0}(f) \cap \overline{W}_{\rho_0}(f) \neq \emptyset$. If $x \in F_{\rho_0}(f) \cap \overline{W}_{\rho_0}(f)$, then the stable manifold of $f$ through $x$ is a horizontal line segment which has a point of tangency with some unstable manifold of $f$.

7. Here we sketch a proof that for $f$ $C^2$ close enough to $L$, $f$ is not structurally stable. We also make some remarks about the $\Omega$-instability of $f$.

Let $\gamma_1$ and $\gamma_2$ be continuous arcs in the plane which have a single point $x$ of intersection. Suppose there is a disk $D$ about $x$ such that $D - \gamma_1$ has two components and $D - \gamma_2$ has two components. Say that the intersection is one-sided if the following is true. If $\gamma_2$ meets the component $V$ of $D - \gamma_1$, then $(\gamma_2 - \{x\}) \cap D \subset V$. Thus we have a picture as in Figure 7.1.

![Figure 7.1](image)

It is clear that if $\gamma_1$ and $\gamma_2$ are smooth arcs with nonvanishing tangent vectors, then a point of one-sided intersection is a point of tangency. In this case, we say that $\gamma_1$ and $\gamma_2$ have a point of one-sided tangency. Notice that if $h$ is a homeomorphism defined in a neighborhood of $x$, then $h(\gamma_1)$ and $h(\gamma_2)$ have a one-sided intersection at $h(x)$. Further, if $\gamma_1$ and $\gamma_2$ have a transversal intersection at $x$, this intersection is not one-sided.
These remarks together with the Kupka-Smale theorem [6, Theorem 6.7] imply that if the stable and unstable manifolds of a hyperbolic fixed point of a diffeomorphism $g$ have a point of one-sided tangency, then $g$ cannot be structurally stable.

Now, by the results of §6, if $f$ is $C^2$ close enough to $L$, $W^s(f)$ has a point of one-sided tangency with $W^u(f)$. Since $W^u(f)$ is completely determined by the action of $f$ on $Q$, a perturbation $g$ of $f$ which agrees with $f$ except in a small neighborhood of $Q_1$ will have $W^u(g) = W^u(f)$. Applying techniques of Smale (I.7 of [6]) to $f|Q_1$, we have that $f$ has a fixed point in $Q$ whose stable manifold contains a dense subset of $W^u(f)$ and whose unstable manifold contains a dense subset of $W^s(f)$.

Therefore, arbitrarily $C^2$ close to $f$, we can find a diffeomorphism $g$ which agrees with $f$ outside a small neighborhood of $Q_1$ and has a fixed point whose stable and unstable manifolds have a one-sided tangency. Since the set of all structurally stable diffeomorphisms is open in $\text{Diff}(S^2)$, we have that $f$ is not structurally stable.

It does not follow immediately from the one-sided tangency of the stable and unstable manifolds of a fixed point of a diffeomorphism $g$ that $g$ is not $C^2$ $\Omega$-stable. However, it is not hard to see that such a $g$ is not $C^1$ $\Omega$-stable, and that if $g_1$ is Kupka-Smale, $C^2$ close to $g$, and $\Omega$-conjugate to $g$, then the conjugating homeomorphism cannot be close to the identity. Furthermore, with a bit more work, it can be shown that, for $r \geq 2$, any $C^r$ diffeomorphism $g$ which is sufficiently $C^2$ close to $L$ is not $C^r\Omega$-stable.

References